

Yukawa's Approach and Dirac's Approach to Relativistic Quantum Mechanics

—Relativistic Harmonic Oscillator Model—

D. HAN and Y. S. KIM*

Systems and Applied Sciences Corporation, Riverdale, Maryland 20840

**Center for Theoretical Physics, Department of Physics and Astronomy
University of Maryland, College Park, Maryland 20742*

(Received March 4, 1980)

It is shown that Dirac's "Poisson bracket" approach to relativistic quantum mechanics is equivalent to the wave function approach which was originally suggested by Yukawa. It is pointed out that this circumstance is very similar to the case of nonrelativistic quantum mechanics in which Heisenberg's commutator formalism is equivalent to the algorithm based on the Schrödinger equation.

§ 1. Introduction

Quantum mechanics and relativity were formulated before most of us were born, and are likely to remain as the two major scientific languages for many years to come. For this reason, the question of whether these two physical theories can be made compatible with each other transcends generations. The basic mathematical apparatus for special relativity is the Poincaré transformation. There are two different but equivalent ways of formulating quantum mechanics. One way is to construct a system of commutators for the operators corresponding to dynamical quantities, and the other way is to construct superposable wave functions from which probability distributions are derived. Thus the attempt to combine quantum mechanics with relativity should take the form of constructing the commutator system and/or wave functions which can be made compatible with transformation properties of special relativity.

As for the wave function method, the most successful approach has been the relativistic oscillator model starting from Yukawa's original work.¹⁾ On the other hand, the most promising commutator formalism was given by Dirac.²⁾ The purpose of the present paper is to show that Dirac's "Poisson bracket" formalism is equivalent to Yukawa's wave function approach. Using the technique of spacetime diagrams, we show first that the starting point of Yukawa is the same as that of Dirac.

We then point out that the ultimate goal of Dirac's approach is to find a spacetime solution of the commutator equations for the generators of the Poincaré group with the subsidiary condition which prevents motions along the time-like

direction. It is shown that the wave functions derivable within the framework of Yukawa's oscillator formalism form the representations of the Poincaré group without time-like excitations. It is shown therefore that the two approaches are equivalent to each other.

In § 2, we show that the physical principles for relativistic quantum mechanics are contained in Eqs. (2), (9) and (10) of Yukawa's original paper.¹¹ It is shown that his equations (2) and (9) represent a combined effect of the space-momentum uncertainty relation of Heisenberg and the "C-number" time-energy uncertainty of Dirac.¹² It is also shown that Yukawa's equation (10) describes the Lorentz deformation property of relativistic wave functions, which takes a natural form in Dirac's light-cone coordinate system.

In § 3, we demonstrate that the relativistic oscillator wave functions derivable in Yukawa's approach form a solution of Dirac's commutator equations. Section 4 deals with the subsidiary condition which reduces the four-dimensional Minkowskian space into a three-dimensional Euclidian space in which nonrelativistic quantum mechanics is valid. It is shown that Dirac's "instant-form" subsidiary condition is equivalent to that of Yukawa which suppresses time-like excitations.

In § 5, the physical implications of §§ 2~4 are discussed. It is pointed out that the wave function method, as in the case of nonrelativistic quantum mechanics, serves many practical purposes in high-energy physics and in physics teaching.

§ 2. Physical basis for Yukawa's approach and Dirac's approach

The purpose of this section is to show that the starting point of Yukawa is identical to that of Dirac. For this purpose, we start with the oscillator model of Yukawa. Let us consider a system of two quarks bound together by a harmonic oscillator of unit strength, and let x_1 and x_2 specify the spacetime coordinates for these two quarks. We are then led to consider the hadronic coordinate X and the internal quark coordinate x defined as

$$\begin{aligned} X &= (x_1 + x_2)/2, \\ x &= (x_1 - x_2)/2\sqrt{2}. \end{aligned} \quad (1)$$

In his original paper,¹¹ Yukawa considered the harmonic oscillator wave function of the form

$$\psi(x) = H_n(z) \exp[-(z^2 + t^2)/2], \quad (2)$$

where z and t are the longitudinal and time-like separations between the two quarks. The transverse components have been suppressed for simplicity. In order to make the theory consistent with relativity and to prevent negative values of (mass)², Yukawa noticed in his equation (9) of his paper¹¹ that time-like excitations should be suppressed, and the form of Eq. (2) reflects this observation.

We can sketch the wave function of Eq. (2) in the spacetime diagram of Fig. 1. According to this figure, there are excitations corresponding to classical

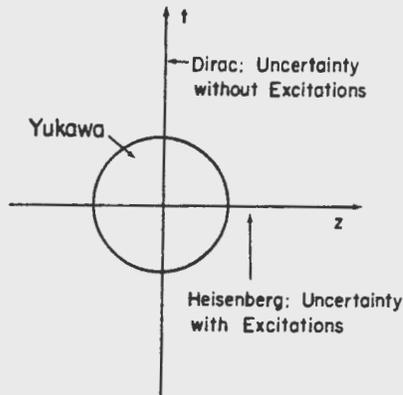


Fig. 1. Spacetime diagram representing Heisenberg's usual space-momentum uncertainty and Dirac's "C-number" time-energy uncertainty relation. There are no excitations along the t axis. This means that there is no Hilbert space associated with this axis. The oscillator wave function of Eq. (2) which was given first by Yukawa combines these two uncertainty relations. This form of wave function was suggested first by Yukawa in Eqs. (2) and (9) of Ref. 1).

motions along the z axis, but there are no motions along the time-separation axis. There is however an uncertainty associated with the ground-state oscillator wave function along this time-like direction.

This spacetime asymmetry is precisely what Dirac observed in his earlier papers.³ He noted that the uncertainty relation between time and energy has to be a "C-number" uncertainty, which is commonly observed in the relation between the energy width and lifetime of unstable systems.⁴ The "C-number" in the matrix language is a one-by-one matrix, and this in turn implies the absence of excited states in the harmonic oscillator.

Let us next discuss special relativity. In Ref. 2), Dirac introduced the light-cone coordinate system in which the variables

$$u = (t+z)/\sqrt{2} \quad \text{and} \quad v = (t-z)/\sqrt{2} \quad (3)$$

play the basic role. The immediate consequence of this choice of variables is that the quantity uv is invariant under Lorentz transformations:

$$uv = (t^2 - z^2)/2 = \text{constant}. \quad (4)$$

This means that the coordinates u and v are elongated and contracted respectively under the Lorentz transformation in such a way that the area of the rectangle in Fig. 2 remains constant.

In Eq. (10) of his paper,¹⁾ Yukawa suggested the form

$$\psi(x, p) = \exp[x^\mu x_\mu/2 - (P \cdot x/M)^2], \quad (5)$$

as a possible Lorentz generalization of the Gaussian factor of Eq. (1), where M and P are the mass and four-momentum of the hadron respectively. In terms of the light-cone variables, this exponential form can be written as

$$\psi(x, P) = \exp\left[-\frac{1}{2}\left(\frac{1-\beta}{1+\beta}u^2 + \frac{1+\beta}{1-\beta}v^2\right)\right], \quad (6)$$

where β is the velocity of the hadron along the z direction. Here again the trivial transverse coordinate variables are suppressed. This formula specified the elliptic region whose major and minor axes are on the light cones and whose area remains invariant under Lorentz transformations. Indeed this is a manifestation of Dirac's light-cone geometry illustrated in Fig. 2.

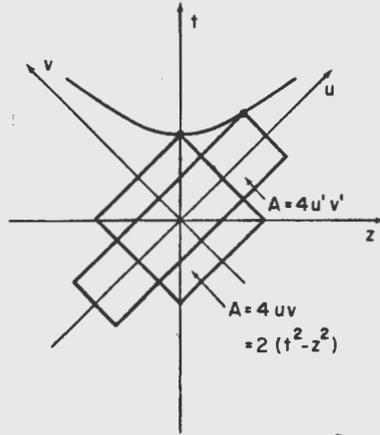


Fig. 2. Spacetime geometry derivable from Dirac's light-cone coordinate system. Under the Lorentz transformation, the square in this figure becomes a rectangle. The area of this rectangle is a Lorentz-invariant quantity. As is indicated in Eq. (6), the covariant form of Eq. (5) which was given by Yukawa in Eq. (10) of Ref. 1) represents this geometry where the square and rectangle are replaced by a circle and an ellipse respectively.

§ 3. Harmonic oscillator solution of Dirac's commutator equations

Thanks to the later works by Markov,⁹⁾ Takabayasi,⁶⁾ Sogami⁷⁾ and Ishida,⁸⁾ Yukawa's oscillator approach had been brought to the form in which the wave functions are solutions of the following Lorentz-invariant oscillator equation:

$$(1/2) [(\partial/\partial X_\mu)^2 + (\partial/\partial x_\mu)^2 - x_\mu^2 + m_0^2] \phi(X, x) = 0 \quad (7)$$

with the subsidiary condition

$$(\partial/\partial X_\mu) a_\mu^\dagger \phi(X, x) = 0, \quad (8)$$

where

$$a_\mu^\dagger = x_\mu + \partial/\partial x^\mu.$$

Using the technique of variable separation, Kim and Noz⁹⁾ obtained explicit solutions to the above equations, and constructed a complete set of wave functions where the form of Eq. (5) is the ground-state wave function. They then started adding quantum mechanical interpretations to the formalism. In order to explain basic high-energy phenomena including mass spectra, form factors, and the parton picture in terms of probability distribution, Kim and Noz⁹⁾ considered also the product of two relativistic wave functions. As was pointed out by Takabayasi,¹⁰⁾ it is an essential step in constructing relativistic quantum mechanics to secure the definition of Poincaré-invariant inner product for any pair of wave functions.

The purpose of this section is to prove that the above-mentioned wave function approach is equivalent to Dirac's "Poisson bracket" formalism, by showing that the physical solutions of the above oscillator differential equation form a spacetime solution of Dirac's commutator equations. In his paper,²⁾ Dirac noted that the generators of the Poincaré transformation form the ten fundamental quantities in the dynamical system. For the present two-body system, these generators take the form

$$\begin{aligned} P_\mu &= -\partial/\partial X^\mu, \\ M_{\mu\nu} &= L_{\mu\nu}^* + L_{\mu\nu}, \end{aligned} \quad (9)$$

where

$$\begin{aligned} L_{\mu\nu}^* &= i[X_\mu\partial/\partial X^\nu - X_\nu\partial/\partial X^\mu], \\ L_{\mu\nu} &= i[x_\mu\partial/\partial x^\nu - x_\nu\partial/\partial x^\mu]. \end{aligned} \quad (10)$$

The operators P_μ generate spacetime translations. $M_{\mu\nu}$ is anti-symmetric under the interchange of μ and ν . Three M_{ij} , with $i, j, = 1, 2, 3$, generate rotations, and three M_{0i} are the generators of Lorentz transformations.

The above ten generators satisfy the following commutations relations:

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [M_{\mu\nu}, P_\rho] &= -g_{\mu\rho}P_\nu + g_{\nu\rho}P_\mu, \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -g_{\mu\rho}M_{\nu\sigma} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\sigma}M_{\rho\nu} + g_{\nu\sigma}M_{\rho\mu}. \end{aligned} \quad (11)$$

Dirac emphasized in his paper²⁾ that *the problem of finding a new dynamical system reduces to the problem of finding a new solution of these equations.* The word "new solution" means a spacetime solution to which proper quantum mechanical interpretation can be given.

It is shown in the literature that we can construct such a spacetime solution from the solutions of the partial differential equation of Eq. (7) in the form

$$\phi(X, x) = \psi(x) \exp[\pm iP \cdot X], \quad (12)$$

where $\psi(x)$ satisfies the harmonic oscillator differential equation

$$H(x)\psi(x) = \lambda\psi(x) \quad (13)$$

with

$$H(x) = (1/2) [(\partial/\partial x_\mu)^2 - x_\mu^2].$$

We are now interested in the solutions of the above oscillator equation which form a solution of Dirac's "Poisson bracket" equations given in Eq. (11). The procedure of finding such solutions is of course to construct the representations of the Poincaré group which are diagonal in the Casimir operators¹¹⁾

$$\begin{aligned} P^2 &= P^\mu P_\mu, \\ W^2 &= W^\mu W_\mu, \end{aligned} \tag{14}$$

where

$$W_\mu = (1/2) \epsilon_{\mu\nu\alpha\beta} P^\nu M^{\alpha\beta}.$$

The above operators commute with the ten generators of the Poincaré group given in Eq. (9).

In their recent paper,¹²⁾ Kim et al. obtained an infinite set of solutions of Eq. (7) which are diagonal in the above Casimir operators, and found the finite subset satisfying the subsidiary condition of Eq. (8) to be

$$\phi(X, x) = \exp(\pm iP \cdot X) \psi(x, P) \tag{15}$$

with

$$\psi(x, P) = (1/\pi)^{1/2} [\exp(-t'^2/2)] R_{nl}(r') Y^m(\theta', \phi'), \tag{16}$$

where

$$t' = (t - \beta z) / (1 - \beta^2)^{1/2},$$

and r', θ', ϕ' are the spherical coordinate variables in a three-dimensional Euclidian space spanned by x, y and z' , where

$$z' = (z - \beta t) / (1 - \beta^2)^{1/2}. \tag{17}$$

R_{nl} in Eq. (16) is the radial wave function for the three-dimensional isotropic harmonic oscillator.

The wave function for the internal quark motion given in Eq. (16) satisfies the subsidiary condition of Eq. (8) which for given P takes the form

$$P^\mu a_\mu \psi(x, P) = 0. \tag{18}$$

This constraint implies that there are no time-like excitations in the Lorentz frame where the hadron is at rest. Classically, this means that there are no motions along this time-like direction. In the quantum system of harmonic oscillators, there is the ground-state wave function associated with Dirac's "C-number" time-energy uncertainty relation.

The oscillator wave function given in Eq. (16) indeed is the physical wave function which serves many useful purposes in high-energy physics. It solves and unifies the oscillator differential equation and the commutator equations given in Eq. (7) and Eq. (11) respectively, subject to the subsidiary condition of Eq. (18). We shall see how Dirac attempted to formulate this subsidiary condition in the following section.

§ 4. Further considerations of the subsidiary condition

In order to construct a geometry for relativistic quantum mechanics,²⁾ Dirac

considered the "instant form" condition

$$x^\mu P_\mu \approx 0, \quad (19)$$

as one of possible constraints which will reduce the four-dimensional Minkowskian space into a three-dimensional Euclidian space in which non-relativistic quantum mechanics is valid. By using the approximate form in Eq. (19), Dirac meant that the equality does not have to be an exact numerical equality, and that it is subject to further physical considerations. What he wanted to do was to "freeze" the motion along the time-separation variable in a manner consistent with quantum mechanics and relativity. If we associate this equality with Dirac's own "C-number" time-energy uncertainty relation which we have discussed in the preceding sections, the constraint of Eq. (19) for the present harmonic oscillator system becomes the subsidiary condition of Eq. (18).

In order that the dynamical system be completely consistent, the subsidiary condition should commute with the generators of the Poincaré group:

$$\begin{aligned} [P_\alpha, P^\mu a_\mu^\dagger] &= 0, \\ [M_{\alpha\beta}, P^\mu a_\mu^\dagger] &= 0. \end{aligned} \quad (20)$$

The above equations follow immediately from the fact that the operator $P^\mu a_\mu^\dagger$ is invariant under translations and Lorentz transformations.

However, this does not complete our discussion of the consistency between the subsidiary condition and the Poincaré transformation, because the operator P^2 is constrained to take the eigenvalues determined by the harmonic oscillator equation given in Eq. (13) through the relation

$$P^2 = \lambda + m_0^2. \quad (21)$$

Therefore, the constraint operator $P^\mu a_\mu^\dagger$ should also commute with $H(x)$ of Eq. (13). However, a simple calculation gives

$$[H(x), P^\mu a_\mu^\dagger] = P^\mu a_\mu^\dagger. \quad (22)$$

This means that the right-hand side is not identically zero, but vanishes only when applied to the wave functions satisfying the subsidiary condition of Eq. (18).

In Ref. 2), Dirac considered also the commutation relations between dynamical quantities and the constraint condition which is "approximately" zero. He asserted that the resulting "Poisson bracket" should also vanish in the same sense. The commutator of Eq. (22) indeed vanishes in accordance with the prescription given by Dirac.

Finally, we have to answer the crucial question of how the harmonic oscillator model can resolve the "real difficulty" which Dirac mentions in connection with the potential term in the "Poisson bracket" formalism. The basic advantage of using the oscillator "potential" is that we can carry out explicit calculations to

interpret the commutators.¹⁰⁾

In formulating his scheme to solve the commutator equations for the generators of the Poincaré group, Dirac²⁾ chose to adopt the view that each constituent particle in "atom" (bound or confined state) is on its mass shell, and that the total energy is the sum of all the free-particle energies of the constituents plus the potential energy. This potential term indeed causes the "real difficulty" in making the commutator system self-consistent.

In the oscillator formalism derivable from Yukawa's original work, we observe the fact that the Casimir operators of the Poincaré group indicate clearly that the mass of the hadron is a Poincaré-invariant constant, but they do not tell anything about the masses of constituent particles. Let us write down the momentum operators of the constituents in terms of the X and x variables:

$$\begin{aligned} p_{1\mu} &= (i/2)\partial/\partial X^\mu + (i/2\sqrt{2})\partial/\partial x^\mu, \\ p_{2\mu} &= (i/2)\partial/\partial X^\mu - (i/2\sqrt{2})\partial/\partial x^\mu. \end{aligned} \quad (23)$$

In order that the constituent mass be a Poincaré-invariant constant, p_1^2 and p_2^2 have to commute with the Casimir operators of Eq. (14) and with the operator $H(x)$ of Eq. (13) which determines the eigenvalues of the Casimir operators. The constituent mass operators derivable from Eq. (23) do not commute with $H(x)$ due to its potential term. We have therefore translated Dirac's real difficulty into

$$[p_1^2, H(x)] \neq 0, \quad [p_2^2, H(x)] \neq 0. \quad (24)$$

These non-vanishing commutators should not cause any difficulty today. The concept of off-mass-shell particles is now firmly established through our experience with Feynman diagrams. The mass of the constituent particle in a bound or confined system does not have to be on its mass shell.

It is extremely interesting to note that the concept of off-mass-shell particles is also derivable from Dirac's time-energy uncertainty relation,¹⁰⁾ which eventually resolves the difficulty he mentioned in Ref. 2). It is also interesting to see that the oscillator formalism derivable from Yukawa's original work enables us to observe this point.

§ 5. Concluding remarks

It is well known that there were two different approaches when the present form of nonrelativistic quantum mechanics was developed more than fifty years ago. They were of course Heisenberg's commutator method and Schrödinger's wave function approach. It is therefore not surprising to find that there have been commutator and wave function approaches to relativistic quantum mechanics. In §§ 2~4, we discussed this aspect and showed that the wave function method starting from Yukawa's 1953 paper¹⁾ is equivalent to the "Poisson bracket" for-

malism of Dirac.²⁾

It is also well known from our experience in nonrelativistic quantum mechanics that the wave function method is more convenient for calculating quantities which can be measured in laboratories. Likewise, the harmonic oscillator wave functions have been very useful in computing measurable quantities in the relativistic quark model. The oscillator formalism has enabled us to construct covariant theoretical models for relativistic extended hadrons to explain the hadronic mass spectra,¹⁴⁾ electromagnetic form factors of the proton,¹⁵⁾ relativistic $SU(6) \otimes O(3)$ model,⁸⁾ the peculiarities in Feynman's parton picture,¹⁶⁾ the jet phenomenon.¹⁷⁾

Because of its mathematical simplicity, the relativistic oscillator model has been very helpful in attacking physical problems. There are indeed many papers on this subject, particularly in this journal. Reference 10) provides an excellent review. There are also several review papers designed for teaching purposes.¹⁸⁾

References

- 1) H. Yukawa, Phys. Rev. **91** (1953), 416.
- 2) P. A. M. Dirac, Rev. Mod. Phys. **21** (1949), 392.
- 3) P. A. M. Dirac, Proc. Roy. Soc. **A114** (1927), 243, 710.
- 4) E. P. Wigner, in *Aspect of Quantum Theory, in Honour of P. A. M. Dirac's 70th Birthday*, edited by A. Salam and E. P. Wigner (Cambridge Univ. Press, London, 1972).
- 5) M. Markov, Nuovo Cim. Suppl. **3** (1956), 760.
- 6) T. Takabayasi, Nuovo Cim. **33** (1964), 668; Prog. Theor. Phys. **34** (1965), 124.
- 7) I. Sogami, Prog. Theor. Phys. **46** (1969), 1352; **50** (1973), 1729.
- 8) S. Ishida, Prog. Theor. Phys. **46** (1971), 1570, 1905.
S. Ishida and J. Otokozaawa, Prog. Theor. Phys. **47** (1972), 2117.
S. Ishida, M. Oda and Y. Yamazaki, Prog. Theor. Phys. **50** (1973), 2000.
- 9) Y. S. Kim and M. E. Noz, Phys. Rev. **D8** (1973), 3521; **12** (1975), 129; Prog. Theor. Phys. **57** (1977), 1373.
- 10) T. Takabayasi, Prog. Theor. Phys. Suppl. No. 67 (1979), 1.
- 11) E. P. Wigner, Ann. Math. **40** (1939), 149.
- 12) Y. S. Kim, M. E. Noz and S. H. Oh, J. Math. Phys. **20** (1979), 1341.
- 13) W. Heitler, *The Quantum Theory of Radiation*, 3rd Ed. (Oxford Univ. Press, London, 1954).
- 14) R. H. Dalitz, in *Proceedings of International Conference on Symmetries and Quark Models*, edited by R. Chand (Gordon and Breach, New York, 1970).
- 15) K. Fujimura, T. Kobayashi and M. Namiki, Prog. Theor. Phys. **43** (1970), 73.
- 16) R. P. Feynman, in *High Energy Collisions, Proceedings of the Third International Conference*, edited by C. N. Yang et al. (Gordon and Breach, New York, 1969).
Y. S. Kim and M. E. Noz, Phys. Rev. **D15** (1977), 335; Prog. Theor. Phys. **60** (1978), 801.
- 17) Y. S. Kim, M. E. Noz and S. H. Oh, Found. of Phys. **9** (1979), 947.
S. H. Oh, Y. S. Kim and M. E. Noz, Found. of Phys. **10** (1980), 635.
- 18) Y. S. Kim and M. E. Noz, Am. J. Phys. **46** (1978), 480, 484.
Y. S. Kim, M. E. Noz and S. H. Oh, Am. J. Phys. **47** (1979), 892.