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Representations of the Poincaré Group for Relativistic Extended Hadrons

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Abstract

Representations of the Poincaré group are constructed from the relativistic harmonic oscillator wave functions which have been effective in describing the physics of internal quark motions in the relativistic quark model. These wave functions are solutions of the Lorentz-invariant harmonic oscillator differential equation in the "cylindrical" coordinate system moving with the hadronic velocity in which the time. separation variable is treated separately. This result enables us to assert that the hadronic mass spectrum is generated by the internal quark level excitation, and that the hadronic spin is due to the internal orbital angular momentum. An addendum relegated to PAPS contains discussions of detailed calculational aspects of the Lorentz transformation, and of solutions of the oscillator equation which are diagonal in the Casimir operators of the homogeneous Lorentz group. It is shown there that the representation of the homogeneous Lorentz group consists of solutions of the oscillator partial differential equation in a "spherical" coordinate system in which the Lorentz-invariant Minkowskian distance between the constituent quarks is the radial variable.

1 Introduction

In building models of relativistic extended hadrons, we have to keep in mind the fundamental fact that the overall space-time symmetry structure is that of the poincaré group [1]. In our previous papers on physical applications of the relativistic harmonic oscillator [2], our primary purpose was to devise a calculational scheme for explaining experimental observations. As was pointed out by Biedenharn *et al.*??, the question of the Poincaré symmetry has not been systematically discussed.

The purpose of the present paper is to address this symmetry problem. We are considering a model hadron consisting of two spinless quarks bound together by a harmonic oscillator potential. In this case, we are led to consider the center-of-mass coordinate which specifies the space-time location of the hadron, and the relative coordinate which specifies the internal space-time separation between the quarks.

Both the hadronic and internal coordinates are subject to Poincaré transformations consisting of translations and Lorentz transformations. The hadronic coordinate undergoes Poincaré transformation in the usual manner. However, the internal coordinate is invariant under translations. This coordinate should, nonetheless, satisfy the poincaré symmetry as a whole. We discuss in this paper the role of this internal coordinate, and show that internal excitations generate the hadronic mass spectrum, and that the internal an- angular momentum corresponds to the spin of the hadron.

In Sec. 2, we formulate the problem using a model hadron consisting of two spinless quarks bound together by a

harmonic oscillator potential of unit strength, and then dis- cuss the generators of the Poincaré group applicable to the entire system. In Sec. 3, we present the oscillator wave functions which are diagonal in the invariant Casimir operators of the Poincaré group.

2 Formulation of the Problem

In our previous papers on physical applications of the relativistic harmonic oscillators, we started with the following Lorentz-invariant differential equation:

$$\left\{2\left[\left(\frac{\partial}{\partial x_1}\right)^2+\left(\frac{\partial}{\partial x_2}\right)^2\right]-\frac{1}{16}\left(x_1-x_2\right)^2+m_0^2\right\}\phi\left(x_1,x_2\right)=0, \qquad (1)$$

where x_1 and x_2 are the space-time coordinates for the two spinless quarks bound together by a harmonic oscillator potential with unit spring constant. In order to simplify the above equation, let us define new coordinate variables

$$X_{=}\frac{x_1 + x_2}{2}, \quad \text{and} \quad x_{=}\frac{x_1 - x_2}{2\sqrt{2}}, \tag{2}$$

The X coordinate represents the space-time specification of the hadron as a whole, while the x variable measures the relative space-time separation between the quarks. In terms of these variables, Eq.(l) can be written as

$$\left\{\left(\frac{\partial}{\partial X}\right)^2 + m_0^2 + \frac{1}{2}\left[\left(\frac{\partial}{\partial x}\right)^2 - x^2\right]\right\}\phi\left(X, x\right) = 0, \tag{3}$$

The above equation is separable in the X and x variables. Thus we write

$$\phi\left(X, x\right) = f\left(X\right)\phi(x), \tag{4}$$

where $f(X)$ and $\psi(x)$ satisfy the following differential equations respectively:

$$\left\{\left(\frac{\partial}{\partial X}\right)^2 + m_0^2 + (\lambda + 1) -\right\} f(X) = 0, \tag{5}$$

and

$$+\frac{1}{2}\left[\left(\frac{\partial}{\partial x}\right)^2 - x^2\right] = (\lambda + 1)\psi(x). \tag{6}$$

The differential equation of Eq.(5) is a Klein Gordon equation, and its solutions are well known. $f(X)$ takes the form

$$f(x) = \exp\left(\pm iP\dot{X}\right) \tag{7}$$

with

$$P^2 = m_0^2 + (\lambda + 1), \tag{8}$$

where P is the four-momentum of the hadron. p^2 is, of course, the $(mass)^2$ of the hadron and is numerically constrained to take the values allowed by Eq.(8). The separation constant λ is determined from the solutions of the harmonic oscillator differential equation of Eq.(6). The physical solutions of the oscillator equation satisfy the subsidiary condition

$$p^{\mu} a_{\mu}^{\dagger} \psi_{\beta}(x) = 0, \tag{9}$$

where

$$a_\mu^\dagger = x_\mu + \frac{\partial}{\partial x^\mu}.$$

The physics of this subsidiary condition has been extensively discussed in the literature [2, 4].

The space-time transformation of the total wave function of Eq.(4) is generated by the following ten generators of the Poincaré group. The operators

$$P_\mu = i\frac{\partial}{\partial X^\mu}. \tag{10}$$

generate space-time translations. Lorentz transformations, which include boosts and rotations, are generated by

$$M_{\mu\nu} = L^*_{\mu\nu} + L_{\mu\nu} \tag{11}$$

where

$$L^*_{\mu\nu} = i\left(X_\mu\frac{\partial}{\partial X^\nu} - X_\nu\frac{\partial}{\partial X^\mu}\right),$$

$$L_{\mu\nu} = i\left(x_\mu\frac{\partial}{\partial x^\nu} - x_\nu\frac{\partial}{\partial x^\mu}\right).$$

The translation operators P_μ, act only on the hadronic coordinate, and do not affect the internal coordinate. The operators $L^*_{\mu\nu}$ and $L_{\mu\nu}$ Lorentz-transform the hadronic and internal coordinates respectively. The above ten generators satisfy the commutation relations for the Poincaré group.

In order to consider irreducible representations of the Poincaré group, we have to construct wave functions which are diagonal in the invariant Casimir operators of the group, which commute with all the generators of Eqs.(10) and (11). The Casimir operators in this case are

$$P^{mu}P_\mu, \quad \text{and} \quad W^\mu W_\mu, \tag{12}$$

where

$$W_\mu = \epsilon_{\mu\nu\alpha\beta}P^\nu M^{\alpha\beta}$$

The eigenvalues of the above P^2 and W^2 represent respectively the mass and sin of the hadron.

3 Physical Wave Functions and Representations of the Poincaré Group

In constructing wave functions diagonal in the Casimir operators of the Poincaré group, we note first that the operator which acts on the wave function in the subsidiary condition of Eq. (9) commutes with these invariant operators:

$$\left[P^2, p^\mu a_\mu^\dagger\right] = 0, \tag{13}$$

$$\left[W^2, p^\mu a_\mu^\dagger\right] = 0. \tag{14}$$

Therefore, the wave functions satisfying the condition of Eq.(9) can be diagonal in the Casimir operators. In order to obtain the solutions explicitly, let us assume without loss of generality that the hadron moves along the z direction with tire velocity parameter β. Then we are led to consider the Lorentz frame where the hadron is at rest, and the coordinate variables are given by

$$x' = x, \qquad y' = y,$$

$$, \qquad z' = \frac{z - \beta t}{\sqrt{1-\beta^2}},$$

$$t' = \frac{t - \beta z}{\sqrt{1-\beta^2}}.$$

The Lorentz-invariant oscillator equation of Eq.(6) is separable in the above variables. In terms of these primed variables, we can construct a complete set of wave functions

$$\psi_\beta(x) = f_b(x) f_s(y) f_n(z') f_k(t'), \tag{15}$$

where

$$f_n(z') = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}} H_n(z') \exp\left(-z'^2/2\right),$$

$$f_k(t') = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}} H_n(z') \exp\left(-t'^2/2\right).$$

If the excitation numbers, $b, ..., k$ are allowed to take all possible nonnegative integer values, the solutions in Eq.(16) form a complete set. However, the eigenvalues λ takes the form

$$\lambda = b + s + n - k. \tag{16}$$

Because the coefficient of $(-k)$ is negative in the above expression, λ has no lower bound, and there is an infinite degeneracy for a given value of λ.

In terms of the primed coordinates, the subsidiary condition of Eq.(9) takes the simple form

$$\left(\frac{\partial}{\partial t'} + t'\right) \psi_\beta(x) = 0. \tag{17}$$

This limits $f_k(t')$ to $f_0(t')$, and the eigenvalue λ becomes

$$\lambda = b + s + n. \tag{18}$$

The physical wave functions satisfying the subsidiary condition of Eq.(9) or (18) have nonnegative values of λ.

As far as the x', y', z' coordinates are concerned, they form an orthogonal Euclidean space, and $f_b(x'), f_s(y'), f_n(x)$ form a complete set in this three-dimensional space. The Hermite polynomials in these Cartesian wave functions can then be combined to form the eigenfunctions of $\mathbf{W}'^2$ which, in terms of the primed coordinate variables, takes the form

$$\mathbf{W}'^2 = M^2 \left(\mathbf{L}'\right)^2, \tag{19}$$

where

$$L_i' = -i\epsilon_{ijk} x_j \frac{\partial}{\partial x_j}.$$

and M is the hadronic mass.

The physical wave functions now take the form

$$\psi_\beta^{\lambda \ell m} = \left[\left(\frac{1}{\pi}\right)^{1/4} \exp\left(-t^2/2\right)\right] R_{\lambda\ell} R(r') Y_{\ell m}\left(\theta', \phi'\right) \tag{20}$$

where r', θ', ϕ' are the radial and spherical variables in the three-dimensional space spanned by x', y', z'. $R_{\lambda\ell}(r')$ is the normalized radial wave function for the three-dimensional isotropic harmonic oscillator, and its form is well known. The above wave function is diagonal in $\mathbf{W}^2$ for which the eigenvalue

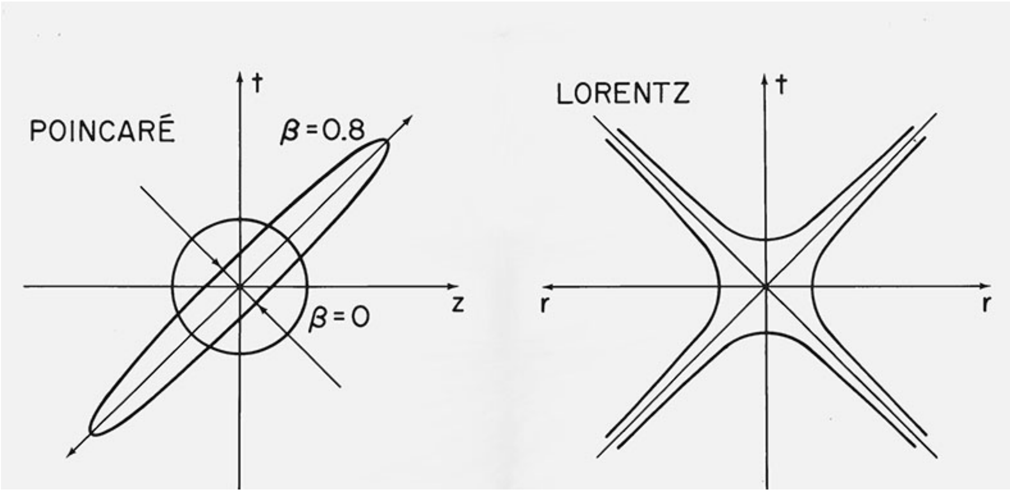

Figure 1: Elliptic and hyperbolic localizations in space-time. The wave functions in the present paper are elliptically localized, and undergo Lorentz deformation as the hadron moves. The Lorentz invariant form $x_\mu x^\mu$ to which we are accustomed, is hyperbolically localized, and is basically different from the form used in the present paper.

is $\ell(\ell + l)M$, and represents the total spin of the hadron in the present case. The quantum number m corresponds to the helicity.

Since the eigenvalue p2 of the Casimir operator P , is constrained to take the numerical values allowed by Eq. (8), the hadronic mass is given by

$$M^2 = m_0^2 + (\lambda + 1). \tag{21}$$

If we relax the subsidiary condition of Eq.(18), we indeed obtain a complete set. In this case, λ of Eq.(17) can become negative for sufficiently large values of k. For $\lambda > 0$, the solutions become

$$\psi_\beta^{\lambda k \ell m} = \left[\left(\frac{1}{\sqrt{\pi}2^k k!}\right)^{1/2} H_k(t) \exp\left(-t^2/2\right)\right] R_{\lambda\ell} R(r') Y_{\ell m}\left(\theta', \phi'\right) \tag{22}$$

For $\lambda < 0$, the solutions take the form

$$\psi_\beta^{\lambda k \ell m} = \left[\left(\frac{1}{\sqrt{\pi}2^{(k-\Lambda)}(k-\lambda)!}\right)^{1/2} H_{k-\lambda}(t) \exp\left(-t^2/2\right)\right] R_{\lambda\ell} R(r') Y_{\ell m}\left(\theta', \phi'\right) \tag{23}$$

The eigenvalues of P^2 and W^2 are again $m_o^2 + (\lambda + 1)$ and $\ell(\ell + l)M^2$ respectively. In both of the above cases, k is allowed to take all possible integer values.

The functional forms of Eqs.(23) and (24) are relatively simple, and they suggest that this representation of the poincaré group corresponds to the solution of the Lorentz-invariant oscillator differential equation in a *cylindrical* coordinate system moving with the hadronic velocity where the t' variable is treated separately. We are then led to the question of why this fact was not known.

Even though the above representations take simple forms, the wave functions contain the following nonconventional features. The first point to note is that they are written as functions of the x', y', z', t' variables. The transverse variables x' and y' are simply x and y respectively. However, z' and t' are linear combinations of z and t. Because the physical meaning of the time-separation variable was not clearly understood, the dependence discouraged us in the past from using it explicitly in representation theory. The explicit use of this variable in the present paper is based on the progress that has been made in our physical understanding of this time-separation variable in terms of measurable quantities, and in terms of the relativistic wave functions carrying a covariant probability interpretation [2].

Another factor which used to discourage the use of the t variable was that we are accustomed to its appearance through the form

$$x_\mu x^\mu = t^2 - r^2, \tag{24}$$

where

$$r^2 = x^2 + y^2 + z^2.$$

In terms of this form, it is very inconvenient, if not impossible, to describe functions which are localized in a finite space-time region. In contrast to the above hyperbolic case, the wave functions which we constructed in this paper are well localized within the region

$$\left(t'^2 + z'^2\right) < 2, \tag{25}$$

due to the Gaussian factor appearing in the wave functions. This elliptic form was obtained from the covariant expression

$$x_\mu x^\mu + 2\left[\frac{x\dot{P}}{M}\right]^2 < 2. \tag{26}$$

The x' and y' variables have been omitted in Eq.(26) because they are trivial. In terms of z and, the above inequality takes the form

$$\left[\frac{1-\beta}{1+\beta}(z+t)^2 + + \frac{1-\beta}{1+\beta}(z-t)^2\right] < 2. \tag{27}$$

We are therefore dealing with the function localized within an elliptic region defined by this inequality, and can control the r variable in the same manner as we do in the case of the spatial variables appearing in nonrelativistic quantum mechanics. This localization property together with the hyperbolic case is illustrated in Fig. 1.

Concluding Remarks

We have shown in this paper that the wave functions used in our previous papers are diagonal in the Casimir operators of the Poincaré group, which specify covariantly the mass and total spin of the hadron. These wave functions are well localized in a space-time region, and undergoes elliptic Lorentz deformation.

An addendum to this paper containing a discussion of Lorentz transformation of the physical wave function and a construction of the representation of the homogeneous Lorentz group is relegated to PAPS [5]. It is shown there that solutions of the oscillator equation diagonal in the Casimir operators of the homogeneous Lorentz group are localized within the Lorentz-invariant hyperbolic region illustrated in Fig. 1.

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[5] See AIP document no. PAPS JMAPA-20-133G12 for twelve pages of discussions of the Lorentz transformation of the physical wave functions, and of the representations of the homogeneous Lorentz group. Order by PAPS number and journal reference from American Institute of Physics, Physics Auxiliary Publication Service, 335 East 45th Street, New York, N.Y. 10017. The price is S1.50 for each microfiche (98 pages), or $5 for photocopies of up to 30 pages with $0.15 for each additional page over 30 pages. Airmail additional. Make checks payable to the American Institute of Physics. This material also appears in Current Physics Microfilm, the monthly microfilm edition of the complete set of journals published by AIP, on the frames immediately following this journal article.

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Addendum to

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PREFACE

This note is an addendum to the authors' article entitled as above, and consists of two sections. The first section (Sec. IV) deals with calculational aspects of the Lorentz transformation of physical wave functions belonging to a representation of the Poincaré group. The second section (Sec. V) contains a detailed analysis of the solutions of the harmonic oscillator wave equation which are diagonal in the Casimir operators of the homogeneous Lorentz group. The representation of the homogeneous Lorentz group consists of solutions of the oscillator partial differential equation in a "spherical" coordinate system in which the Lorentz-invariant Minkowskian distance between the constituent quarks is the radial variable. The calculations presented in Sec. V, together with those of Sec. III, illustrate the point that the Poincaré group is quite different from a direct product of a translation and the homogeneous Lorentz group.

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IV. TRANSFORMATION PROPERTIES OF THE PHYSICAL WAVE FUNCTIONS

In Sec. III, we used the z axis as the direction of the hadronic velocity. As is well known, this choice is purely a matter of convenience. Since the four momentum p determines the direction and magnitude of the hadronic velocity, we can generalize the expression $\psi_\beta(x)$ to $\psi(x,p)$, contending that a "z-axis" can be assigned to the four momentum p. In this section, we are interested in the transformation which will change $\psi(x,p)$ to $\psi(x,p')$:

$$\psi(x,p') = T(p',p)\,\psi(x,p) \; . \tag{29}$$

In order to find the operator $T(p',p)$, we define first the operator which will transform the zero-velocity wave function to $\psi(x,p)$:

$$\psi(x,p) = T(p)\,\psi_o(x) \; . \tag{30}$$

Next, we define the rotation operator $R(p',p)$ which will rotate the zero-velocity wave function from the direction specified by p to that of p' around an axis perpendicular to the two velocities defined by p and p' respectively. Then according to the procedure due to Wigner,[1]

$$\psi(x,p') = T(p')R(p',p)T^{-1}(p)\,\psi(x,p) \; . \tag{31}$$

Thus

$$T(p',p) = T(p')R(p',p)T^{-1}(p) \; . \tag{32}$$

The rotation matrix $R(p',p)$ is finite and unitary, and it takes the form

$$R(p',p) = \exp[-i\,\xi\,\vec{n}\cdot\vec{L}] \; , \tag{33}$$

where $\vec{n}$ is the direction of the rotation axis, and ξ is the angle of rotation. Thus the remaining problem is to construct the boost operator $T(p)$. The procedure of constructing this operator is also well established,[5] and $T(p)$ can be written as

$$T(p) = \exp\,[-i\,\eta\,\vec{k}\cdot\vec{K}] \quad . \tag{34}$$

where $\vec{k}$ is the direction of the boost, and

$$\sinh \eta = \beta/(1-\beta^2)^{1/2} \; . \tag{35}$$

The boost generators K_i take the form

$$K_i = -i\left(x_i \frac{\partial}{\partial t} + t\,\frac{\partial}{\partial x_i}\right), \tag{36}$$

where i = 1, 2, 3.

The wave function $\psi(x,p)$ can be obtained from repeated applications of the boost generators along the $\vec{k}$ direction on the zero-velocity wave function. Thus we are interested in the application of the K_i operators on the $\beta = 0$ wave functions given in Eqs.(23) and (24). As in the case of the rotation operator, it is more convenient to work with K_3 and $K_{\pm}$, where

$$K_{\pm} = K_1 \pm iK_2 \quad . \tag{37}$$

If we apply these operators to the wave functions of Eqs.(23) and (24) with $\beta = 0$,

$$\begin{aligned} iK_3\psi^{\lambda k}_{\ell m} &= A_3 \left[\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right]^{1/2} Y^m_{\ell+1}\,(\theta,\phi) \\ &\quad + B_3 \left[\frac{(\ell+1)(\ell-m)}{(2\ell+1)(2\ell-1)}\right]^{1/2} Y^m_{\ell-1}\,(\theta,\phi) \quad , \\ iK_{\pm}\psi^{\lambda k}_{\ell m} &= A_{\pm} \left[\frac{(\ell\pm m+1)(\ell\pm m+2)}{(2\ell+1)(2\ell+3)}\right]^{1/2} Y^{m\pm 1}_{\ell+1}\,(\theta,\phi) \\ &\quad + B_{\pm} \left[\frac{(\ell\mp m)(\ell\mp m-1)}{(2\ell+1)(2\ell-1)}\right]^{1/2} Y^{m\mp 1}_{\ell-1}\,(\theta,\phi) \; . \end{aligned} \tag{38}$$

The notation in the above expression is slightly different from that of Eqs.(23) and (24), but this difference should not cause any confusion. The coefficients of the spherical harmonics given in the above formulas have been calculated by Naimark.[5] What is new in the present work is that the cofficients A and B can now be calculated. They take the form

$$
\begin{aligned}
A_3 &= Q_{-\ell} F_\ell^{\lambda k}(r,t), \qquad & B_3 &= Q_{\ell+1} F_\ell^{\lambda k}(r,t), \\
A_\pm &= \mp\, Q_{-\ell} F_\ell^{\lambda k}(r,t), \qquad & B_\pm &= \pm\, Q_{\ell+1} F_\ell^{\lambda k}(r,t), \\
\text{where } Q_\ell &= t \frac{\partial}{\partial r} + r \frac{\partial}{\partial t} + \ell \frac{t}{r}, \\
F_\ell^{\lambda k}(r,t) &= \psi_{\ell m}^{\lambda k}(x) / Y_\ell^m(\theta,\phi) .
\end{aligned}
\tag{39}
$$

In order to understand the nature of the Lorentz transformation T(p) more precisely let us next concentrate our efforts on the case where the boost is along the z direction. The form of K_3 applied to $\psi_{\ell m}^{\lambda k}$ in Eq.(38) indicates that the helicity quantum number m is conserved under this boost transformation. However, the transformed wave function contains all possible values of ℓ . This is a reflection of the following non-vanishing commutator:

$$[(\vec{L})^2, (\vec{L}')^2] = 0 \quad . \tag{40}$$

The forms of A_3 and B_3 in Eq.(39) indicate that the transformed wave function will contain radial, orbital and time-like wave functions which are different from those given initially. In order to see this and other points more specifically, let us consider a finite boost along the z direction of the physical wave function given in Eq.(21) with $\beta = 0$.

Because only the z and t components are affected by the boost along the z direction, we have to rewrite the wave function of Eq.(21) in terms of the

Cartesian variables and their Hermite polynomials. The portion of the wave function affected by this transformation is

$$\psi_o^{n,o}(z,t) = \left(\frac{1}{\pi 2^n n!}\right)^{1/2} H_n(z)\exp[-\tfrac{1}{2}(z^2+t^2)]. \tag{41}$$

The superscript o indicates that there are no time-like excitations. Let us now consider the transformation

$$\psi_\beta^{n,o}(z,t) = T(p)\psi_o^{n,o}(z,t) , \tag{42}$$

$$\psi_\beta^{n,o}(z,t) = \psi_o^{n,o}(z',t') , \tag{43}$$

and ask what T(p) does on $\psi_o^{n,o}(z,t)$. In order to answer this question, we write Eq.(42) as

$$\psi_\beta^{n,o}(z,t) = \sum_{n',k'} A_{n',k'}^{n,o}(\beta)\psi_o^{n',k'}(z,t) . \tag{44}$$

It is shown in Ref. 2 (Am. J. Phys. 46, 480) that this expression can be simplified to

$$\psi_\beta^{n,o}(z,t) = \sum_k^\infty A_k^n(\beta)\psi_o^{n+k,k}(z,t) . \tag{45}$$

The remaining problem now is to determine the coefficient $A_k^n(\beta)$. For this purpose, we note that

$$\begin{aligned} A_k^n(\beta) &= \int dzdt\ \psi_\beta^{n,o}(z,t)\ \psi_o^{n+k,k}(z,t) \\ &= \frac{1}{\pi}\left(\frac{1}{2}\right)^n \left\{\frac{1}{2^k n!(n+k)!}\right\}^2 \\ &\quad \times \int dzdt\ H_{n+k}(z)\ H_k(t)\ H_n(z') \\ &\qquad \times \exp[-\tfrac{1}{2}(z^2+z'^2+t^2+t'^2)] . \end{aligned} \tag{46}$$

In the above integral, the Hermite polynomials and the Gaussian form are mixed with the kinematics of Lorentz transformation. However, if we use the generating function for the Hermite polynomial as Ruiz did in his paper,[6] this integral can be evaluated easily, and

$$A_k^n(\beta) = (1-\beta^2)^{\frac{n+1}{2}} \beta^k \left[\frac{(n+k)!}{n!k!}\right]^{1/2} . \quad (47)$$

Thus Eq.(44) can be writen as

$$\psi_\beta^{n,o}(z,t) = (\tfrac{1}{\pi})^{1/2} (\tfrac{1}{2})^{n/2} (1-\beta^2)^{n+1/2}$$
$$\times \left[\sum_{k=0}^{\infty} \frac{\beta^k}{2^k k!} H_{n+k}(z)H_k(t)\right]\exp[-\tfrac{1}{2}(z^2+t^2)]. \quad (48)$$

Let us now examine the implications of the above results. Since the expression in Eq.(48) requires a sum over the longitudinal excitations equal to or higher than n, the Lorentz transformed wave functions with a given value in the moving frame is a sum over all corresponding ℓ values of the wave functions at rest. This result is not inconsistent with $K_3\psi$ of Eq.(38) and the non-vanishing commutator of Eq.(40).

In the hadronic rest system, the wave function with $k > 0$ is not a physical solution. Therefore, the wave function of Eq.(48) is a sum of nonphysical solutions in the rest frame. However, after the summation, these nonphysical wave functions in the rest frame form a physical wave function corresponding to a moving hadron with velocity parameter β. The wave function of Eq.(48) indeed satisfies the subsidiary condition of Eq.(18).

In this section, we restricted our discussion to the physical wave functions $\psi_\beta^{n,o}(x)$. For other wave functions $\psi_\beta^{n,k}$ with $k > 0$, the calculation becomes more complicated. However, the mathematics is essentially the same.

V. REPRESENTATIONS OF THE HOMOGENEOUS LORENTZ GROUP

While the primary purpose of this paper is to study representations of the Poincaré group, it is of interest to see how they are different from those of the homogeneous Lorentz group. Since the internal coordinate x is not affected by translations, we can construct solutions which represent the homogenous Lorentz group generated by the $L_{\mu\nu}$ of Eq.(11). The solutions representing this group are diagonal in the Casimir operators which commute with the $L_{\mu\nu}$:

$$C_1 = (1/2)\, L^{\mu\nu} L_{\mu\nu} \quad ,$$
$$C_2 = (1/4)\, \varepsilon_{\mu\nu\alpha\beta} L^{\mu\nu} L^{\alpha\beta} \quad . \tag{49}$$

The fact that these Casimir operators are different from those of the Poincaré group has been emphasized by many authors.[7] However, since the representation of the Poincaré group leads to a specific way in which the partial differential equation given in Eq.(6) is separated, we are natually interested here in the coordinate system for obtaining wave functions diagonal in the Casimir operators of Eq.(49).

In terms of the rotation and boost generators, C_1 and C_2 can be written as

$$C_1 = \vec{L}^{\,2} - \vec{K}^{\,2} \quad ,$$
$$C_2 = \vec{L} \cdot \vec{K} \quad . \tag{50}$$

If we calculate C_2 using the explicit expressions for L and K, then for this spinless case

$$C_2 = \vec{L} \cdot \vec{K} = 0. \tag{51}$$

Thus we are led to consider only C_1. In order to construct solutions diagonal in this operator, we use the "spherical" coordinate system in which the "radius" is Lorentz invariant:

$$\rho = \{|t^2 - r^2|\}^{1/2} \tag{52}$$

with

$$r = (x^2 + y^2 + z^2)^{1/2} ,$$

$$t = \pm \rho \cosh\alpha , \tag{53}$$

$$r = |\rho \sinh\alpha|$$

for $|t| > r$, and

$$t = \rho \sinh\alpha, \tag{54}$$

$$r = \rho \cosh\alpha$$

for $|t| < r$. For both cases, we use the usual three-dimensional spherical coordinate for x, y, z.

$$x = r \sin\theta \cos\phi,$$

$$y = r \sin\theta \sin\phi, \tag{55}$$

$$z = r \cos\theta .$$

In terms of ρ, α, θ, ϕ, the differential equation of Eq.(6) takes the form

$$\frac{1}{\rho^3} \frac{\partial}{\partial\rho} (\rho^3 \frac{\partial\psi}{\partial\rho}) + [\frac{1}{\rho^2} (\vec{K}^2 - \vec{L}^2) - \rho^2]\psi = \varepsilon\psi , \tag{56}$$

where $\varepsilon = \pm 2(\lambda+1)$ for the time-like and space-like cases respectively. The form of $\vec{L}$ is well known. The operator $(\vec{L}^2 - \vec{K}^2)$ takes the form

$$(\vec{L}^2 - \vec{K}^2) = \frac{1}{\sinh^2\alpha} \frac{\partial}{\partial\alpha} (\sinh^2\alpha \frac{\partial}{\partial\alpha}) - \frac{1}{\sinh^2\alpha} \vec{L}^2 \tag{57}$$

for $|t| > r$, and

$$(\vec{L}^2 - \vec{K}^2) = \frac{1}{\cosh^2\alpha} (\frac{\partial}{\partial\alpha} \cosh^2\alpha \frac{\partial}{\partial\alpha}) + \frac{1}{\cosh^2\alpha} \vec{L}^2 \quad (58)$$

for $|t| < r$. We are interested in the solutions which are diagonal in the above operators.

In order to construct this representation, we solve the partial differential equation given in Eq.(56) by separating the variables:

$$\psi(x) = R(\rho)\ B(\alpha,\theta,\phi) \ . \quad (59)$$

In terms of $R(\rho)$ and $B(\alpha,\theta,\phi)$, Eq.(56) is separated into

$$[\frac{1}{\rho^3} \frac{\partial}{\partial\rho} (\rho^3 \frac{\partial}{\partial\rho}) - \frac{\eta}{\rho^2} + \rho^2 - \varepsilon]R(\rho) = 0 \ , \quad (60)$$

and

$$(\vec{L}^2 - \vec{K}^2)B(\alpha,\theta,\phi) = \eta\ B(\alpha,\theta,\phi) \ . \quad (61)$$

In order that the radial equation have regular solutions,

$$\eta = n(n+1), \quad n = 1, 2, 3, --- \quad . \quad (62)$$

The radial wave function in this case takes the form

$$R^{\ell}_{\mu,n}(\rho) = \rho^n L_{\mu}^{(n+1)}(\rho^2) \exp (-\frac{\rho^2}{2}). \quad (63)$$

with $\varepsilon = 2(2\mu + n)$, $\mu = 0, 1, 2, \ldots$. $L_{\mu}^{(n+1)}(\rho^2)$ is the generalized Laguerre function.[8]

With this preparation, we now write the "angular" function B as

$$B^{\ell}_{n}(\alpha,\theta,\phi) = A^{\ell}_{n}(\alpha)Y^{m}_{\ell}(\theta,\phi) \ . \quad (64)$$

For the time-like region where $|t| > r$, we use the notation:

$$A_n^{\ell}(\alpha) = T_n^{\ell}(\alpha) \quad , \tag{65}$$

and for the space-like region,

$$A_n^{\ell}(\alpha) = S_n^{\ell}(\alpha) \quad . \tag{66}$$

Then $T_n^{\ell}(\alpha)$ and $S_n^{\ell}(\alpha)$ satisfy the following differential equations respectively.

$$\frac{\partial}{\partial\alpha}\,(\sinh^2\alpha\; T_n^{\ell}) - [n(n+2) + \ell(\ell+1)]\; T_n^{\ell} = 0 \quad , \tag{67}$$

$$\frac{\partial}{\partial\alpha}\,(\cosh^2\alpha\; S_n^{\ell}) - [n(n+2) - \ell(\ell+1)]\; S_n^{\ell} = 0 \quad . \tag{68}$$

If $\ell = 0$, the solutions to the above equation take the form

$$T_n^{o}(\alpha) = \frac{\sinh\,(n+1)\,\alpha}{\sinh\,\alpha} \quad ,$$

$$S_n^{o}(\alpha) = \frac{\cosh\,(n+1)\,\alpha}{\cosh\,\alpha} \quad . \tag{69}$$

For non-vanishing values of ℓ,

$$T_n^{\ell}(\alpha) = (\sinh\,\alpha)^{\ell}\,\left(\frac{1}{\sinh\,\alpha}\,\frac{d}{d\alpha}\right)^{\ell}\,T_n^{o}(\alpha) \quad ,$$

$$S_n^{\ell}(\alpha) = (\cosh\,\alpha)^{\ell}\,\left(\frac{1}{\cosh\,\alpha}\,\frac{d}{d\alpha}\right)^{\ell}\,S_n^{o}(\alpha) \quad . \tag{70}$$

The solutions given in Eqs.(69) and (70) become infinite when $\alpha \to \infty$. This means that the Lorentz harmonics are singular along the light cones. At this point, we are tempted to make n imaginary in order to make $T_n(\alpha)$ and $S_n(\alpha)$ normalizable. In fact, this and other mathematically interesting possibilities have been extensively discussed in the literature.[9] However, if n takes non-integer values, the radial wave function becomes singular along the light cones.

In either case, the singularity along the light cones is unavoidable.

The wave functions which are diagonal in the Casimir operator C_1 take the form

$$\psi^{\mu,n}_{\ell,m}(x) = R^{n}_{\mu}(\rho)\ A^{\ell}_{n}(\alpha)\ Y^{m}_{\ell}(\theta,\phi) \ , \tag{71}$$

where R^{n}_{μ}, and A^{ℓ}_{n} are given in Eqs.(63) and (65,66) respectively. The localization property of the above solution is dictated by the Gaussian factor in the radial function $R^{n}_{\mu}(\rho)$, and is illustrated by the hyperbolas in Fig. 1. Unlike the case of the wave functions representing the Poincaré group, the localization region in this case is independent of the hadronic velocity and is thus Lorentz invariant.

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