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Addendum to

REPRESENTATIONS OF THE POINCARÉ GROUP FOR RELATIVISTIC EXTENDED HADRONS

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PREFACE

This note is an addendum to the authors' article entitled as above, and consists of two sections. The first section (Sec. IV) deals with calculational aspects of the Lorentz transformation of physical wave functions belonging to a representation of the Poincaré group. The second section (Sec. V) contains a detailed analysis of the solutions of the harmonic oscillator wave equation which are diagonal in the Casimir operators of the homogeneous Lorentz group. The representation of the homogeneous Lorentz group consists of solutions of the oscillator partial differential equation in a "spherical" coordinate system in which the Lorentz-invariant Minkowskian distance between the constituent quarks is the radial variable. The calculations presented in Sec. V, together with those of Sec. III, illustrate the point that the Poincaré group is quite different from a direct product of a translation and the homogeneous Lorentz group.

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IV. TRANSFORMATION PROPERTIES OF THE PHYSICAL WAVE FUNCTIONS

In Sec. III, we used the z axis as the direction of the hadronic velocity. As is well known, this choice is purely a matter of convenience. Since the four momentum p determines the direction and magnitude of the hadronic velocity, we can generalize the expression $\psi_\beta(x)$ to $\psi(x,p)$, contending that a "z-axis" can be assigned to the four momentum p. In this section, we are interested in the transformation which will change $\psi(x,p)$ to $\psi(x,p')$:

$$\psi(x,p') = T(p',p)\ \psi(x,p) \ . \tag{29}$$

In order to find the operator $T(p',p)$, we define first the operator which will transform the zero-velocity wave function to $\psi(x,p)$:

$$\psi(x,p) = T(p)\ \psi_o(x) \ . \tag{30}$$

Next, we define the rotation operator $R(p',p)$ which will rotate the zero-velocity wave function from the direction specified by p to that of p' around an axis perpendicular to the two velocities defined by p and p' respectively. Then according to the procedure due to Wigner,[1]

$$\psi(x,p') = T(p')R(p',p)T^{-1}(p)\ \psi(x,p) \ . \tag{31}$$

Thus

$$T(p',p) = T(p')R(p',p)T^{-1}(p) \ . \tag{32}$$

The rotation matrix $R(p',p)$ is finite and unitary, and it takes the form

$$R(p',p) = \exp[-i\,\xi\,\vec{n}\cdot\vec{L}] \ , \tag{33}$$

where $\vec{n}$ is the direction of the rotation axis, and ξ is the angle of rotation. Thus the remaining problem is to construct the boost operator $T(p)$. The procedure of constructing this operator is also well established,[5] and $T(p)$ can be written as

$$T(p) = \exp\,[-i\,\eta\,\vec{k}\cdot\vec{K}] \quad . \tag{34}$$

where $\vec{k}$ is the direction of the boost, and

$$\sinh\eta = \beta/(1-\beta^2)^{1/2} \, . \tag{35}$$

The boost generators K_i take the form

$$K_i = -i\left(x_i \frac{\partial}{\partial t} + t\frac{\partial}{\partial x_i}\right), \tag{36}$$

where i = 1, 2, 3.

The wave function $\psi(x,p)$ can be obtained from repeated applications of the boost generators along the $\vec{k}$ direction on the zero-velocity wave function. Thus we are interested in the application of the K_i operators on the $\beta = 0$ wave functions given in Eqs.(23) and (24). As in the case of the rotation operator, it is more convenient to work with K_3 and $K_\pm$, where

$$K_\pm = K_1 \pm iK_2 \quad . \tag{37}$$

If we apply these operators to the wave functions of Eqs.(23) and (24) with $\beta = 0$,

$$\begin{aligned}
iK_3\psi^{\lambda k}_{\ell m} &= A_3\left[\frac{(\ell+m+1)(\ell-m+1)}{(2\ell+1)(2\ell+3)}\right]^{1/2} Y^m_{\ell+1}(\theta,\phi) \\
&\quad + B_3\left[\frac{(\ell+1)(\ell-m)}{(2\ell+1)(2\ell-1)}\right]^{1/2} Y^m_{\ell-1}(\theta,\phi) \quad , \\
iK_\pm\psi^{\lambda k}_{\ell m} &= A_\pm\left[\frac{(\ell\pm m+1)(\ell\pm m+2)}{(2\ell+1)(2\ell+3)}\right]^{1/2} Y^{m\pm 1}_{\ell+1}(\theta,\phi) \\
&\quad + B_\pm\left[\frac{(\ell\mp m)(\ell\mp m-1)}{(2\ell+1)(2\ell-1)}\right]^{1/2} Y^{m\mp 1}_{\ell-1}(\theta,\phi) \quad .
\end{aligned} \tag{38}$$

The notation in the above expression is slightly different from that of Eqs.(23) and (24), but this difference should not cause any confusion. The coefficients of the spherical harmonics given in the above formulas have been calculated by Naimark.[5] What is new in the present work is that the cofficients A and B can now be calculated. They take the form

$$\begin{aligned}
A_3 &= Q_{-\ell} F_\ell^{\lambda k}(r,t), \qquad & B_3 &= Q_{\ell+1} F_\ell^{\lambda k}(r,t), \\
A_\pm &= \mp Q_{-\ell} F_\ell^{\lambda k}(r,t), \qquad & B_\pm &= \pm Q_{\ell+1} F_\ell^{\lambda k}(r,t),
\end{aligned}$$

$$\text{where } Q_\ell = t\frac{\partial}{\partial r} + r\frac{\partial}{\partial t} + \ell\frac{t}{r}, \qquad (39)$$

$$F_\ell^{\lambda k}(r,t) = \psi_{\ell m}^{\lambda k}(x)/Y_\ell^m(\theta,\phi) .$$

In order to understand the nature of the Lorentz transformation T(p) more precisely let us next concentrate our efforts on the case where the boost is along the z direction. The form of K_3 applied to $\psi_{\ell m}^{\lambda k}$ in Eq.(38) indicates that the helicity quantum number m is conserved under this boost transformation. However, the transformed wave function contains all possible values of ℓ . This is a reflection of the following non-vanishing commutator:

$$[(\vec{L})^2, (\vec{L}')^2] = 0 \quad . \qquad (40)$$

The forms of A_3 and B_3 in Eq.(39) indicate that the transformed wave function will contain radial, orbital and time-like wave functions which are different from those given initially. In order to see this and other points more specifically, let us consider a finite boost along the z direction of the physical wave function given in Eq.(21) with $\beta = 0$.

Because only the z and t components are affected by the boost along the z direction, we have to rewrite the wave function of Eq.(21) in terms of the

Cartesian variables and their Hermite polynomials. The portion of the wave function affected by this transformation is

$$\psi_o^{n,o}(z,t) = \left(\frac{1}{\pi 2^n n!}\right)^{1/2} H_n(z)\exp[-\frac{1}{2}(z^2+t^2)]. \tag{41}$$

The superscript o indicates that there are no time-like excitations. Let us now consider the transformation

$$\psi_\beta^{n,o}(z,t) = T(p)\psi_o^{n,o}(z,t) , \tag{42}$$

$$\psi_\beta^{n,o}(z,t) = \psi_o^{n,o}(z',t') , \tag{43}$$

and ask what T(p) does on $\psi_o^{n,o}(z,t)$. In order to answer this question, we write Eq.(42) as

$$\psi_\beta^{n,o}(z,t) = \sum_{n',k'} A^{n,o}_{n',k'}(\beta)\psi_o^{n',k'}(z,t) . \tag{44}$$

It is shown in Ref. 2 (Am. J. Phys. 46, 480) that this expression can be simplified to

$$\psi_\beta^{n,o}(z,t) = \sum_k^\infty A_k^n(\beta)\psi_o^{n+k,k}(z,t) . \tag{45}$$

The remaining problem now is to determine the coefficient $A_k^n(\beta)$. For this purpose, we note that

$$\begin{aligned} A_k^n(\beta) &= \int dzdt\ \psi_\beta^{n,o}(z,t)\ \psi_o^{n+k,k}(z,t) \\ &= \frac{1}{\pi}\left(\frac{1}{2}\right)^n \left\{\frac{1}{2^k n!(n+k)!}\right\}^2 \\ &\quad \times \int dzdt\ H_{n+k}(z)\ H_k(t)\ H_n(z') \\ &\quad \times \exp\left[-\frac{1}{2}(z^2+z'^2+t^2+t'^2)\right] . \end{aligned} \tag{46}$$

In the above integral, the Hermite polynomials and the Gaussian form are mixed with the kinematics of Lorentz transformation. However, if we use the generating function for the Hermite polynomial as Ruiz did in his paper,[6] this integral can be evaluated easily, and

$$A_k^n(\beta) = (1-\beta^2)^{\frac{n+1}{2}} \beta^k \left[\frac{(n+k)!}{n!k!}\right]^{1/2} . \tag{47}$$

Thus Eq.(44) can be writen as

$$\psi_\beta^{n,o}(z,t) = \left(\frac{1}{\pi}\right)^{1/2} \left(\frac{1}{2}\right)^{n/2} (1-\beta^2)^{n+1/2}$$
$$\times \left[\sum_{k=0}^{\infty} \frac{\beta^k}{2^k k!} H_{n+k}(z)H_k(t)\right]\exp\left[-\frac{1}{2}(z^2+t^2)\right]. \tag{48}$$

Let us now examine the implications of the above results. Since the expression in Eq.(48) requires a sum over the longitudinal excitations equal to or higher than n, the Lorentz transformed wave functions with a given value in the moving frame is a sum over all corresponding ℓ values of the wave functions at rest. This result is not inconsistent with $K_3\psi$ of Eq.(38) and the non-vanishing commutator of Eq.(40).

In the hadronic rest system, the wave function with $k > 0$ is not a physical solution. Therefore, the wave function of Eq.(48) is a sum of nonphysical solutions in the rest frame. However, after the summation, these nonphysical wave functions in the rest frame form a physical wave function corresponding to a moving hadron with velocity parameter β. The wave function of Eq.(48) indeed satisfies the subsidiary condition of Eq.(18).

In this section, we restricted our discussion to the physical wave functions $\psi_\beta^{n,o}(x)$. For other wave functions $\psi_\beta^{n,k}$ with $k > 0$, the calculation becomes more complicated. However, the mathematics is essentially the same.

V. REPRESENTATIONS OF THE HOMOGENEOUS LORENTZ GROUP

While the primary purpose of this paper is to study representations of the Poincaré group, it is of interest to see how they are different from those of the homogeneous Lorentz group. Since the internal coordinate x is not affected by translations, we can construct solutions which represent the homogenous Lorentz group generated by the $L_{\mu\nu}$ of Eq.(11). The solutions representing this group are diagonal in the Casimir operators which commute with the $L_{\mu\nu}$:

$$C_1 = (1/2)\, L^{\mu\nu} L_{\mu\nu} \quad ,$$
$$C_2 = (1/4)\, \varepsilon_{\mu\nu\alpha\beta} L^{\mu\nu} L^{\alpha\beta} \quad . \tag{49}$$

The fact that these Casimir operators are different from those of the Poincaré group has been emphasized by many authors.[7] However, since the representation of the Poincaré group leads to a specific way in which the partial differential equation given in Eq.(6) is separated, we are natually interested here in the coordinate system for obtaining wave functions diagonal in the Casimir operators of Eq.(49).

In terms of the rotation and boost generators, C_1 and C_2 can be written as

$$C_1 = \vec{L}^{\,2} - \vec{K}^{\,2} \quad ,$$
$$C_2 = \vec{L} \cdot \vec{K} \quad . \tag{50}$$

If we calculate C_2 using the explicit expressions for L and K, then for this spinless case

$$C_2 = \vec{L} \cdot \vec{K} = 0. \tag{51}$$

Thus we are led to consider only C_1. In order to construct solutions diagonal in this operator, we use the "spherical" coordinate system in which the "radius" is Lorentz invariant:

$$\rho = \{|t^2-r^2|\}^{1/2} \tag{52}$$

with

$$r = (x^2 + y^2 + z^2)^{1/2} ,$$

$$t = \pm \rho \cosh \alpha , \tag{53}$$

$$r = |\rho \sinh \alpha|$$

for $|t| > r$, and

$$t = \rho \sinh \alpha, \tag{54}$$

$$r = \rho \cosh \alpha$$

for $|t| < r$. For both cases, we use the usual three-dimensional spherical coordinate for x, y, z.

$$x = r \sin\theta \cos\phi,$$

$$y = r \sin\theta \sin\phi, \tag{55}$$

$$z = r \cos\theta .$$

In terms of ρ, α, θ, ϕ, the differential equation of Eq.(6) takes the form

$$\frac{1}{\rho^3} \frac{\partial}{\partial\rho} (\rho^3 \frac{\partial\psi}{\partial\rho}) + [\frac{1}{\rho^2} (\vec{K}^2 - \vec{L}^2) - \rho^2]\psi = \varepsilon \psi , \tag{56}$$

where $\varepsilon = \pm 2(\lambda+1)$ for the time-like and space-like cases respectively. The form of $\vec{L}$ is well known. The operator $(\vec{L}^{\,2} - \vec{K}^{\,2})$ takes the form

$$(\vec{L}^2 - \vec{K}^2) = \frac{1}{\sinh^2\alpha} \frac{\partial}{\partial\alpha} (\sinh^2\alpha \frac{\partial}{\partial\alpha}) - \frac{1}{\sinh^2\alpha} \vec{L}^2 \tag{57}$$

for $|t| > r$, and

$$(\vec{L}^2 - \vec{K}^2) = \frac{1}{\cosh^2\alpha} \left(\frac{\partial}{\partial\alpha} \cosh^2\alpha \frac{\partial}{\partial\alpha}\right) + \frac{1}{\cosh^2\alpha} \vec{L}^2 \tag{58}$$

for $|t| < r$. We are interested in the solutions which are diagonal in the above operators.

In order to construct this representation, we solve the partial differential equation given in Eq.(56) by separating the variables:

$$\psi(x) = R(\rho)\ B(\alpha,\theta,\phi)\ . \tag{59}$$

In terms of $R(\rho)$ and $B(\alpha,\theta,\phi)$, Eq.(56) is separated into

$$\left[\frac{1}{\rho^3} \frac{\partial}{\partial\rho} \left(\rho^3 \frac{\partial}{\partial\rho}\right) - \frac{\eta}{\rho^2} + \rho^2 - \varepsilon\right]R(\rho) = 0\ , \tag{60}$$

and

$$(\vec{L}^2 - \vec{K}^2)B(\alpha,\theta,\phi) = \eta\ B(\alpha,\theta,\phi)\ . \tag{61}$$

In order that the radial equation have regular solutions,

$$\eta = n(n+1), \quad n = 1,\ 2,\ 3,\ --- \quad . \tag{62}$$

The radial wave function in this case takes the form

$$R^{\ell}_{\mu,n}(\rho) = \rho^n L^{(n+1)}_{\mu}(\rho^2) \exp\left(-\frac{\rho^2}{2}\right). \tag{63}$$

with $\varepsilon = 2(2\mu + n)$, $\mu = 0, 1, 2, \ldots$. $L^{(n+1)}_{\mu}(\rho^2)$ is the generalized Laguerre function.[8]

With this preparation, we now write the "angular" function B as

$$B^{\ell}_{n}(\alpha,\theta,\phi) = A^{\ell}_{n}(\alpha)Y^{m}_{\ell}(\theta,\phi)\ . \tag{64}$$

For the time-like region where $|t| > r$, we use the notation:

$$A_n^\ell(\alpha) = T_n^\ell(\alpha) \quad , \tag{65}$$

and for the space-like region,

$$A_n^\ell(\alpha) = S_n^\ell(\alpha) \ . \tag{66}$$

Then $T_n^\ell(\alpha)$ and $S_n^\ell(\alpha)$ satisfy the following differential equations respectively.

$$\frac{\partial}{\partial\alpha} (\sinh^2\alpha \; T_n^\ell) - [n(n+2) + \ell(\ell+1)] \; T_n^\ell = 0 \quad , \tag{67}$$

$$\frac{\partial}{\partial\alpha} (\cosh^2\alpha \; S_n^\ell) - [n(n+2) - \ell(\ell+1)] \; S_n^\ell = 0 \ . \tag{68}$$

If $\ell = 0$, the solutions to the above equation take the form

$$T_n^o(\alpha) = \frac{\sinh\,(n+1)\,\alpha}{\sinh\,\alpha} \quad ,$$

$$S_n^o(\alpha) = \frac{\cosh\,(n+1)\,\alpha}{\cosh\,\alpha} \ . \tag{69}$$

For non-vanishing values of ℓ,

$$T_n^\ell(\alpha) = (\sinh\,\alpha)^\ell \left(\frac{1}{\sinh\,\alpha}\frac{d}{d\alpha}\right)^\ell T_n^o(\alpha) \ ,$$

$$S_n^\ell(\alpha) = (\cosh\,\alpha)^\ell \left(\frac{1}{\cosh\,\alpha}\frac{d}{d\alpha}\right)^\ell S_n^o(\alpha) \ . \tag{70}$$

The solutions given in Eqs.(69) and (70) become infinite when $\alpha \to \infty$. This means that the Lorentz harmonics are singular along the light cones. At this point, we are tempted to make n imaginary in order to make $T_n(\alpha)$ and $S_n(\alpha)$ normalizable. In fact, this and other mathematically interesting possibilities have been extensively discussed in the literature.[9] However, if n takes non-integer values, the radial wave function becomes singular along the light cones.

In either case, the singularity along the light cones is unavoidable.

The wave functions which are diagonal in the Casimir operator C_1 take the form

$$\psi^{\mu,n}_{\ell,m}(x) = R^n_\mu(\rho)\ A^\ell_n(\alpha)\ Y^m_\ell(\theta,\phi) \ , \tag{71}$$

where R^n_μ, and A^ℓ_n are given in Eqs.(63) and (65,66) respectively. The localization property of the above solution is dictated by the Gaussian factor in the radial function $R^n_\mu(\rho)$, and is illustrated by the hyperbolas in Fig. 1. Unlike the case of the wave functions representing the Poincaré group, the localization region in this case is independent of the hadronic velocity and is thus Lorentz invariant.

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