

GROUP CONTRACTIONS: INONU, WIGNER, AND EINSTEIN

Y.S. KIM*

*Department of Physics, University of Maryland,
College Park, Maryland 20742, U.S.A.*

Received 11 October 1996

Einstein's $E = mc^2$ unifies the momentum-energy relation for massive and massless particles. According to Wigner, the internal space-time symmetries of massive and massless particles are isomorphic to $O(3)$ and $E(2)$ respectively. According to Inonu and Wigner, $O(3)$ can be contracted to $E(2)$ in the large-radius limit. It is noted that the $O(3)$ -like little group for massive particles can be contracted to the $E(2)$ -like little group for massless particles in the limit of large momentum and/or small mass. It is thus shown that transverse rotational degrees of freedom for massive particles become contracted to gauge degrees of freedom for massless particles.

1. Introduction

In 1953¹ Inonu and Wigner published their classic paper on group contractions. The concept is very simple. When we walk on the surface of the earth, we are making Euclidean transformations on a two-dimensional plane: translations and rotations around an axis perpendicular to the plane. Strictly speaking, however, we are making rotations around the center of the earth with a very large radius. The work of Inonu and Wigner was based on this geometry.

In their paper, Inonu and Wigner were interested in unitary representations of $E(2)$, and consequently they were interested in mathematics in which the spherical harmonics become Bessel functions. In this report, we point out that non-unitary finite dimensional representations of $E(2)$ are also physically relevant and occupy an important place in physics as well as in mathematics.³ While Inonu and Wigner were interested only in the plane tangential to the sphere with a large radius, we would like to point out here that there is a cylinder tangential to the sphere. We shall point out also that the motion of a point on the cylindrical surface could be formulated in terms of an $E(2)$ -like group.^{4,5,6}

Let us now get to the main issue. In 1939,² Wigner observed that internal space-time symmetries of relativistic particles are dictated by their respective little groups.² The little group is the maximal subgroup of the Lorentz group which leaves the four-momentum of the particle invariant. He showed that the little groups for massive and massless particles are isomorphic to the three-dimensional rotation group and the two-dimensional Euclidean group respectively. Wigner's 1939 paper

*E-mail: kim@umdhep.umd.edu

indeed gives a covariant picture massive particles with spins, and connects the helicity of a massless particle with the rotational degree of freedom in the group $E(2)$. This paper also gives many homework problems for us to solve. In this report, let us concentrate ourselves on the following two questions.

- First, like the three-dimensional rotation group, $E(2)$ is a three-parameter group. It contains two translational degrees of freedom in addition to the rotation. What physics is associated with the translational-like degrees of freedom for the case of the $E(2)$ -like little group?
- Second, as is shown by Inonu and Wigner,¹ the rotation group $O(3)$ can be contracted to $E(2)$. Does this mean that the $O(3)$ -like little group can become the $E(2)$ -like little group in a certain limit?

As for the first question, it has been shown by various authors that the translation-like degrees of freedom in the $E(2)$ -like little group is the gauge degree of freedom for massless particles.⁷ As for the second question, it is not difficult to guess that the $O(3)$ -like little group becomes the $E(2)$ -like little group in the limit of large momentum/mass.⁸ However, the non-trivial result is that the transverse rotational degrees of freedom become gauge degrees of freedom.⁹ We can compare this result to Einstein's energy-momentum relation $E = (m^2 + p^2)^{1/2}$, which gives two different formulas for massive and massless particles. The following table summarizes this comparison.

Table 1. Covariance of the energy-momentum relation, and covariance of the internal space-time symmetry groups.

Massive, Slow	COVARIANCE	Massless, Fast
$E = p^2/2m$	Einstein's $E = mc^2$	$E = cp$
S_3	Wigner's Little Group	S_3
S_1, S_2		Gauge Trans.

In Sec. 2, we shall discuss three-dimensional geometry of a sphere in detail. One of the three axes can become both contracted or expanded. It is shown in Sec. 3 that the two different deformations discussed in Sec. 2 lead to Lorentz boost in the light-cone coordinate system. The little groups are discussed in the light-cone coordinate system, and the contraction of the $O(3)$ -like little group to the $E(2)$ -like little group is calculated directly from the two different deformations of the sphere. In Sec. 4, we discuss possible applications of the contraction procedure to other physical phenomena.

2. Three-dimensional Geometry of the Little Groups

The little groups for massive and massless particles are isomorphic to $O(3)$ and $E(2)$ respectively. It is not difficult to construct the $O(3)$ -like geometry of the little

group for a massive particle at rest.² The generators L_i of the rotation group satisfy the commutation relations:

$$[L_i, L_j] = i\epsilon_{ijk}L_k. \quad (1)$$

Transformations applicable to the coordinate variables x, y , and z are generated by

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2)$$

The Euclidean group $E(2)$ is generated by L_3, P_1 and P_2 , with

$$P_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \quad (3)$$

and they satisfy the commutation relations:

$$[P_1, P_2] = 0, \quad [L_3, P_1] = iP_2, \quad [L_3, P_2] = -iP_1. \quad (4)$$

The generator L_3 is given in Eq.(2). When applied to the vector space $(x, y, 1)$, P_1 and P_2 generate translations on in the xy plane. The geometry of $E(2)$ is also quite familiar to us.

Let us transpose the above algebra. Then P_1 and P_2 become Q_1 and Q_2 , where

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \quad (5)$$

respectively. Together with L_3 , these generators satisfy the same set of commutation relations as that for L_3, P_1 , and P_2 given in Eq.(4)

$$[Q_1, Q_2] = 0, \quad [L_3, Q_1] = iQ_2, \quad [L_3, Q_2] = -iQ_1. \quad (6)$$

These matrices generate transformations of a point on a circular cylinder. Rotations around the cylindrical axis are generated by L_3 . The Q_1 and Q_2 matrices generate the transformation:

$$\exp(-i\xi Q_1 - i\eta Q_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \xi & \eta & 1 \end{pmatrix}. \quad (7)$$

When applied to the space (x, y, z) , this matrix changes the value of z while leaving the x and y variables invariant.⁴ This corresponds to a translation along the cylindrical axis. The J_3 matrix generates rotations around the axis. We shall call the group generated by J_3, Q_1 and Q_2 the *cylindrical group*.

We can achieve the contractions to the Euclidean and cylindrical groups by taking the large-radius limits of

$$P_1 = \frac{1}{R}B^{-1}L_2B, \quad P_2 = -\frac{1}{R}B^{-1}L_1B, \quad (8)$$

and

$$Q_1 = -\frac{1}{R}BL_2B^{-1}, \quad Q_2 = \frac{1}{R}BL_1B^{-1}, \quad (9)$$

where

$$B(R) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R \end{pmatrix}.$$

The vector spaces to which the above generators are applicable are $(x, y, z/R)$ and (x, y, Rz) for the Euclidean and cylindrical groups respectively. They can be regarded as the north-pole and equatorial-belt approximations of the spherical surface respectively.

Since $P_1(P_2)$ commutes with $Q_2(Q_1)$, we can consider the following combination of generators.

$$F_1 = P_1 + Q_1, \quad F_2 = P_2 + Q_2. \quad (10)$$

Then these operators also satisfy the commutation relations:

$$[F_1, F_2] = 0, \quad [L_3, F_1] = iF_2, \quad [L_3, F_2] = -iF_1. \quad (11)$$

However, we cannot make this addition using the three-by-three matrices for P_i and Q_i to construct three-by-three matrices for F_1 and F_2 , because the vector spaces are different for the P_i and Q_i representations. We can accommodate this difference by creating two different z coordinates, one with a contracted z and the other with an expanded z , namely $(x, y, Rz, z/R)$. Then the generators become four-by-four matrices, and F_1 and F_2 take the form

$$F_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (12)$$

The rotation generator L_3 is also a four-by-four matrix:

$$L_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (13)$$

These four-by-four matrices satisfy the $E(2)$ -like commutation relations of Eq.(11).

Next, let us consider the transformation matrix generated by the above matrices. It is easy to visualize the transformations generated by P_i and Q_i . It would be easy to visualize the transformation generated by F_1 and F_2 , if P_i commuted with Q_i . However, P_i and Q_i do not commute with each other, and the transformation matrix takes a somewhat complicated form:

$$\exp \{-i(\xi F_1 + \eta F_2)\} = \begin{pmatrix} 1 & 0 & 0 & \xi \\ 0 & 1 & 0 & \eta \\ \xi & \eta & 1 & (\xi^2 + \eta^2)/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (14)$$

3. Little Groups in the Light-cone Coordinate System

Let us now study the group of Lorentz transformations using the light-cone coordinate system. If the space-time coordinate is specified by (x, y, z, t) , then the light-cone coordinate variables are (x, y, u, v) for a particle moving along the z direction, where

$$u = (z + t)/\sqrt{2}, \quad v = (z - t)/\sqrt{2}. \quad (15)$$

The transformation from the conventional space-time coordinate to the above system is achieved through a similarity transformation.

It is straight-forward to write the rotation generators J_i and boost generators K_i in this light-cone coordinate system.⁵ If a massive particle is at rest, its little group is generated by J_1, J_2 and J_3 . For a massless particle moving along the z direction, the little group is generated by N_1, N_2 and J_3 , where

$$N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1, \quad (16)$$

which can be written in the matrix form as

$$N_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (17)$$

and J_3 takes the form of the four-by-four matrix given in Eq.(13)

These matrices satisfy the commutation relations:

$$[J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1, \quad [N_1, N_2] = 0. \quad (18)$$

Let us go back to F_1 and F_2 of Eq.(12). Indeed, they are proportional to N_1 and N_2 respectively. Since F_1 and F_2 are somewhat simpler than N_1 and N_2 , and since the commutation relations of Eq.(18) are invariant under multiplication of N_1 and N_2 by constant factors, we shall hereafter use F_1 and F_2 for N_1 and N_2 .

In the light-cone coordinate system, the boost matrix takes the form

$$B(R) = \exp(-i\rho K_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & 1/R \end{pmatrix}, \quad (19)$$

with $\rho = \ln(R)$, and $R = \sqrt{(1 + \beta)/(1 - \beta)}$, where β is the velocity parameter of the particle. The boost is along the z direction. Under this transformation, x and y coordinates are invariant, and the light-cone variables u and v are transformed as

$$u' = Ru, \quad v' = v/R. \quad (20)$$

If we boost J_2 and J_1 and multiply them by $\sqrt{2}/R$, as

$$W_1(R) = -\frac{\sqrt{2}}{R} B J_2 B^{-1} = \begin{pmatrix} 0 & 0 & -i/R^2 & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ i/R^2 & 0 & 0 & 0 \end{pmatrix},$$

$$W_2(R) = \frac{\sqrt{2}}{R} B J_1 B^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i/R^2 & i \\ 0 & i & 0 & 0 \\ 0 & i/R^2 & 0 & 0 \end{pmatrix}, \quad (21)$$

then $W_1(R)$ and $W_2(R)$ become F_1 and F_2 of Eq.(12) respectively in the large- R limit.

The most general form of the transformation matrix is

$$D(\xi, \eta, \alpha) = D(\xi, \eta, 0)D(0, 0, \alpha), \quad (22)$$

with

$$D(\xi, \eta, 0) = \exp \{-i(\xi F_1 + \eta F_2)\}, \quad D(0, 0, \alpha) = \exp(-i\alpha J_3). \quad (23)$$

The matrix $D(0, 0, \alpha)$ represents a rotation around the z axis. In the light-cone coordinate system, $D(\xi, \eta, 0)$ takes the form of Eq.(14). It is then possible to decompose it into

$$D(\xi, \eta, 0) = C(\xi, \eta)E(\xi, \eta)S(\xi, \eta), \quad (24)$$

where

$$\begin{aligned} S(\xi, \eta) &= I + \frac{1}{2}[C(\xi, \eta), E(\xi, \eta)] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & (\xi^2 + \eta^2)/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ E(\xi, \eta) &= \exp(-i\xi P_1 - i\eta P_2) = \begin{pmatrix} 1 & 0 & 0 & \xi \\ 0 & 1 & 0 & \eta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ C(\xi, \eta) &= \exp(-i\xi Q_1 - i\eta Q_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \xi & \eta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (25)$$

Let us consider the application of the above transformation matrix to an electromagnetic four-potential of the form

$$A^\mu(x) = A^\mu e^{i(kz - \omega t)}, \quad (26)$$

with

$$A^\mu = (A_1, A_2, A_u, A_v), \quad (27)$$

where $A_u = (A_3 + A_0)/\sqrt{2}$, and $A_v = (A_3 - A_0)/\sqrt{2}$. If we impose the Lorentz condition, the above four-vector becomes

$$A^\mu = (A_1, A_2, A_u, 0), \quad (28)$$

The matrix $S(\xi, \eta)$ leaves the above four-vector invariant. The same is true for the $E(\xi, \eta)$ matrix. Both $E(\xi, \eta)$ and $S(\xi, \eta)$ become identity matrices when applied to

four-vectors with vanishing fourth component. Thus only the $C(\xi, \eta)$ matrix performs non-trivial operations. As in the case of Eq.(7), it performs transformations parallel to the cylindrical axis, which in this case is the direction of the photon momentum. It leaves the transverse components of the four vector invariant, but changes the longitudinal and time-like components at the same rate. This is a gauge transformation.

It is remarkable that the algebra of Lorentz transformations given in this section can be explained in terms of the geometry of deformed spheres developed in Sec. 2.

4. Outlook

In this report, we discussed the group contraction procedure initiated by Inonu and Wigner in 1953. We have seen how this contraction procedure can be applied to the question of unifying the internal space-time symmetries of massive and massless particles. The result is summarized in Table 1 given in Sec. 1.

Indeed, many papers have been published on this subject since then, and it is likely that there will be many more in the future. This is the reason why we are having this conference. Algebraically speaking, the contraction is the process of obtaining one set of Lie algebra to another set through a limiting process. Geometrically speaking, the contraction is a transition from one surface to another through a tangential area. We are of course talking about groups with rich representations. Indeed, the concept of group contraction generates very rich mathematics.

The question then is whether there are other physical phenomena which can be fit into the framework defined in Table 1. In 1955,¹⁰ Hofstadter and MacAllister reported their experimental result showing that the proton is not a point particle but has a space-time extension. In 1964,¹¹ Gell-Mann proposed the quark model in which the proton is a bound state of the constituent particles called quarks. The bound states so constructed is now called hadrons.

On the other hand, in 1969,¹² Feynman observed that the proton coming from a high-energy accelerator consists of "partons" with properties quite different from those of the quarks inside the hadron. These days, the partons are routinely regarded as the quarks. However, it is not trivial to show why they have thoroughly different properties. In order to solve this puzzle, we need a model of covariant bound states which can be Lorentz-transformed.

Even before Hofstadter discovered the non-zero size of the proton, Yukawa formulated a relativistic theory of extended hadrons based on the covariant harmonic oscillators in 1953.¹³ Yukawa's idea was later extended to explain the quark model by Feynman, Kislinger, and Ravndal.¹⁴ Yukawa's initial formalism was then extended to accommodate Wigner's $O(3)$ -like little group for massive particles.¹⁵

If we use this covariant oscillator formalism, it is possible to show that the quark and parton models are two different manifestations of the same covariant entity,¹⁶ just as in the case of Einstein's energy-momentum relation and Wigner's little group. The result is illustrated in the following table.

Table 2. Addition of the quark-parton covariance to the space-time symmetry table of Table 1.

Massive, Slow	COVARIANCE	Massless, Fast
$E = p^2/2m$	Einstein's $E = mc^2$	$E = cp$
S_3	Wigner's Little Group	S_3
S_1, S_2		Gauge Trans.
Quark Model	Covariant Oscillators	Parton Model

References

1. E. Inonu and E. P. Wigner, *Proc. Natl. Acad. Sci. (U.S.A.)* **39**, 510 (1953).
2. E. P. Wigner, *Ann. Math.* **40**, 149 (1939).
3. Y. S. Kim and M. E. Noz, *Theory and Applications of the Poincaré Group* (Reidel, Dordrecht, 1986).
4. Y. S. Kim and E. P. Wigner, *J. Math. Phys.* **28**, 1175 (1987).
5. Y. S. Kim and E. P. Wigner, *J. Math. Phys.* **31**, 55 (1990).
6. For an earlier discussion of the cylindrical geometry in connection with Maxwell's equations, see L. J. Boya and J. A. de Azcarraga, *An. R. Soc. Esp. Fis. y Quim.* **A63**, 143 (1967).
7. A. Janner and T. Janssen, *Physica* **53**, 1 (1971); *ibid.* **60**, 292 (1972); J. Kuperzstych, *Nuovo Cimento* **31B**, 1 (1976); D. Han and Y. S. Kim, *Am. J. Phys.* **49**, 348 (1981); J. J. van der Bij, H. van Dam and Y. J. Ng, *Physica* **116A**, 307 (1982); D. Han, Y. S. Kim and D. Son, *Phys. Rev.* **D26**, 3717 (1982).
8. S. P. Misra and J. Maharana, *Phys. Rev.* **D14**, 133 (1976); S. Ferrara and C. Savoy, in *Supergravity 1981*, S. Ferrara and J. G. Taylor, eds. (Cambridge Univ. Press, Cambridge, 1982), p. 151; D. Han, Y. S. Kim and D. Son, *J. Math. Phys.* **27**, 2228 (1986); P. Kwon and M. Villasante, *J. Math. Phys.* **29**, 560 (1988); *ibid.* **30**, 201 (1989).
9. D. Han, Y. S. Kim and D. Son, *Phys. Lett.* **131B**, 327 (1983).
10. R. Hofstadter and R. W. McAllister, *Phys. Rev.* **98**, 217 (1955).
11. M. Gell-Mann, *Phys. Lett.* **13**, 598 (1964).
12. R. P. Feynman, in *High Energy Collisions*, Proceedings of the Third International Conference, Stony Brook, New York, C. N. Yang *et al.*, eds. (Gordon and Breach, New York, 1969); J. D. Bjorken and E. A. Paschos, *Phys. Rev.* **185**, 1975 (1969).
13. H. Yukawa, *Phys. Rev.* **91**, 416 (1953).
14. R. P. Feynman, M. Kislinger and F. Ravndal, *Phys. Rev.* **D3**, 2706 (1971).
15. Y. S. Kim, M. E. Noz and S. H. Oh, *J. Math. Phys.* **20**, 1341 (1979).
16. Y. S. Kim and M. E. Noz, *Phys. Rev.* **D15**, 335 (1977); Y. S. Kim, *Phys. Rev. Lett.* **63**, 348 (1989).