Dirac's Harmonic Oscillators

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Paul A. M. Dirac is known to us through the Dirac equation for spin-1/2 particles. But his main interest was in the foundational problems. First, Dirac was never satisfied with the probabilistic formulation of quantum mechanics. Second, if we tentatively accept the present form of quantum mechanics, he was insisting that it has to be consistent with special relativity. He wrote several important papers on this subject. Let us look at some of his papers on this subject.

During World War II, he was looking into constructing representations of the Lorentz group using harmonic oscillator wave functions [2]. The Lorentz group is the language of special relativity, and the present form of quantum mechanics starts with harmonic oscillators. Presumably, therefore, he was interested in making quantum mechanics Lorentz-covariant by constructing representations of the Lorentz group using harmonic oscillators.

In his 1945 paper, Dirac considers the Gaussian

$$\exp\left\{-\frac{1}{2}\left(x^2 + y^2 + z^2 + t^2\right)\right\}. \tag{1}$$

We note that this Gaussian form is in the $(x,\ y,\ z,\ t)$ coordinate variables. Thus, if we consider Lorentz boost along the z direction, we can drop x and y variables, and write the above equation as

$$\exp\left\{-\frac{1}{2}\left(z^2 + t^2\right)\right\}. \tag{2}$$

This is a strange expression for those who believe in Lorentz invariance. The expression

$$\exp\left\{-\frac{1}{2}\left(z^2 - t^2\right)\right\}. \tag{3}$$

is invariant, but Dirac's Gaussian form of Eq.(2) is not. On the other hand, this expression is consistent with his earlier papers on the time-energy uncertainty relation [1]. In those papers, Dirac observes that there is a time-energy uncertainty relation, while there are no excitations along the time axis. He called this "c-number time-energy uncertainty" relation. When I was talking with Dirac in 1978, he clearly mentioned this word again. He said further that his is one of the stumbling block in combining quantum mechanics with relativity. This situation is illustrated in Fig. 1.

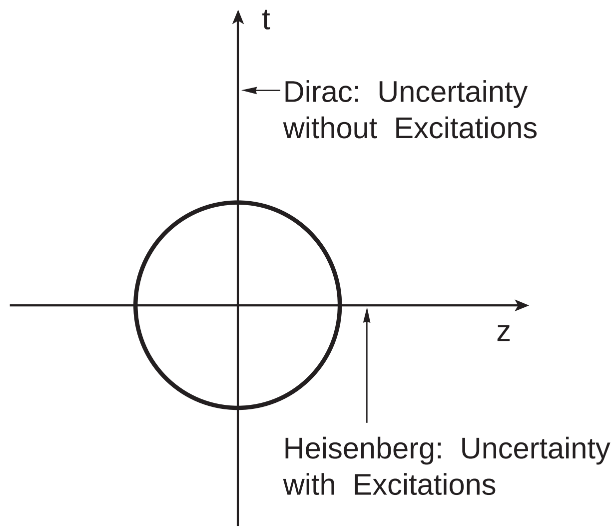

Figure 1: Space-time picture of quantum mechanics. There are quantum excitations along the space-like longitudinal direction, but there are no excitations along the time-like direction. The time-energy relation is a c-number uncertainty relation.

Let us look at Fig. 1 carefully. This figure is a pictorial representation of Dirac's Eq.(2), with localization in both space and time coordinates. Then Dirac's fundamental question would be how to make this figure covariant? This is where Dirac stops. However, this is not the end of the Dirac story.

In 1949, the Reviews of Modern Physics published a special issue to celebrate Einstein's 70th birthday. This issue contains Dirac paper entitled " Forms of Relativistic Dynamics." In this paper, he introduced his light-cone coordinate system, in which Lorentz-boost becomes a squeeze transformation.

When the system is boosted along the z direction, the transformation takes the form

$$\begin{pmatrix} z' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}. \tag{4}$$

This is not a rotation, and people still feel strange about this form of transformation. In 1949, Dirac introduced his light-cone variables defined as [3]

$$u = (z + t)/\sqrt{2}, \qquad v = (z - t)/\sqrt{2}, \tag{5}$$

the boost transformation of Eq.(4) takes the form

$$u' = e^{\eta}u, \qquad v' = e^{-\eta}v. \tag{6}$$

The u variable becomes expanded while the v variable becomes contracted, as is illustrated in Fig. 2.

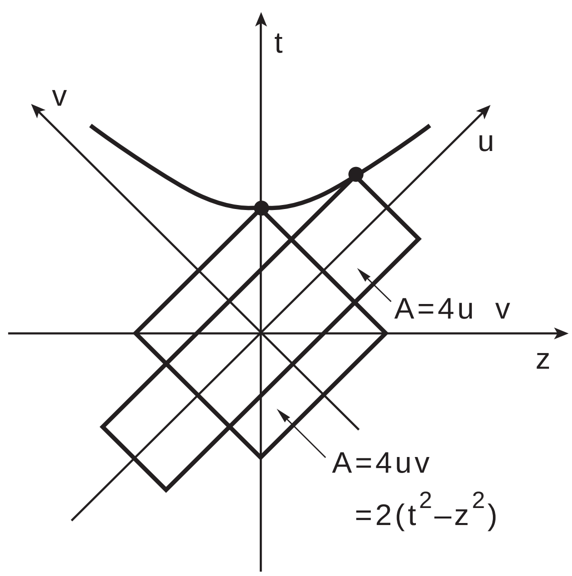

Figure 2: Lorentz boost in the light-cone coordinate system.

If we combine Fig. 1 and Fig. 2, then we end up with Fig. 3.

In mathematical formulas, this transformation changes the Gaussian form of Eq.(2) into

$$\psi_{\eta}(z,t) = \exp\left\{-\frac{1}{2}\left(e^{-2\eta}u^2 + e^{2\eta}v^2\right)\right\}. \tag{7}$$

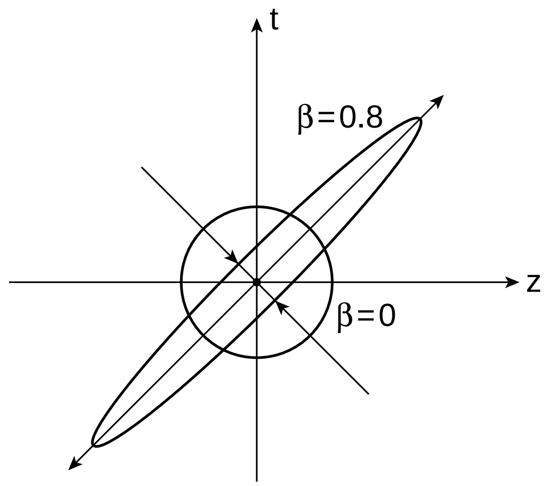

Figure 3: Effect of the Lorentz boost on the space-time wave function. The circular space-time distribution in the rest frame becomes Lorentz-squeezed to become an elliptic distribution.

This form becomes (2) when η becomes zero. The transition from Eq.(2) to Eq.(7) is a squeeze transformation.

Since 1973, I wrote many papers starting from this circle/ellipse figure mostly with Marilyn Noz. Indeed, this the starting point of my research program. These days, this circle/ellipse figure is widely known as the logo for the International Conference on Squeezed States and Uncertainty Relations. This conference is primarily for those working on quantum optics and foundations of quantum mechanics. If I meet those people, many of them greet me by saying "How is the squeezed state?" They say this because of the circle/ellipse logo.

Dirac's interest in harmonic oscillator did not stop with his 1945. In his 1962 [4] paper, he constructed a representation of the $O(3,2)$ deSitter group using two coupled harmonic oscillators. This paper contains not only the mathematics of combining quantum mechanics, but also forms the foundations of two-mode squeezed states which are so essential modern quantum optics.

References

[1] P. A. M. Dirac, Proc. Roy. Soc. (London) **A114**, 243 and 710 (1927).

[2] P. A. M. Dirac, Proc. Roy. Soc. (London) **A183**, 284 (1945).

[3] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[4] P. A. M. Dirac, J. Math. Phys. **4**, 524 (1962).