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Covariant Forms of the Uncertainty Relations

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It is pointed out that the covariant harmonic oscillator formalism combines covariantly Heisenberg's uncertainty relation and the minimal time-energy uncertainty which Wigner suggested in 1972 in order to explain the well-known relation between the life time and the energy width of resonance states. It is then shown that this combined uncertainty relation leads to the peculiarities observed in Feynman's parton phenomenon.

I. INTRODUCTION

The present form of quantum mechanics which was formulated fifty years ago is based on Heisenberg's uncertainty principle and is most conveniently represented by Schrödinger's superposable wave functions. While nonrelativistic quantum mechanics continued to serve useful purposes in atomic and nuclear physics, two most important questions were left unanswered. The first question was and still is whether we should be satisfied by the statistical interpretation of quantum mechanics. [1] The second question is whether the probabilistic interpretation which is inherent in the present form of quantum mechanics can accommodate special relativity.

While the first question is still important and interesting, the present author is not able to add anything new to the existing literature. [1] This naturally leads us to the second question. If we settle with the probabilistic interpretation, how can we then make quantum mechanics covariant?

There have been many concentrated efforts to answer this second question. One very important step made toward this direction was of course the covariant perturbation theory developed by Feynman, Schwinger, and Tomonaga which achieved some spectacular successes in quantum electrodynamics. [2] However, the present form of quantum field theory is not concerned with the question of constructing localized probability distribution which is so crucial in interpreting the uncertainty relation. [3]

In constructing a covariant localized probability distribution, the unavoidable question is whether there is a time-energy uncertainty relation which will be linearly combined with the longitudinal uncertainty when the entire system undergoes a Lorentz transformation. In this connection, we note the word "minimal time-energy uncertainty relation" which Wigner used to describe the well-known uncertainty relation between the life time and the energy width of resonance states while being consistent with the absence of time-like excitations in nature. [4]

The purpose of the present paper is to show that Feynman's parton phenomenon [5] is derivable from this form of time-energy uncertainty relation. In Sec. II, it is pointed out that the covariant harmonic oscillator is the natural language to represent the minimal time-energy uncertainty relation. In Sec. III, we give a space-time picture of hadronic Lorentz deformation. In Sec. IV, we use a diagramatic language to show that Feynman's parton phenomenon is a manifestation of the minimal time-energy uncertainty relation.

II. COVARIANT HARMONIC OSCILLATORS AND THE MINIMAL TIME-ENERGY UNCERTAINTY RELATION

Because of its mathematical simplicity, the harmonic oscillator has been very useful in interpreting Heisenberg's original form of the uncertainity principle. It is therefore not unnatural to expect that a Gaussian form of the harmonic-oscillator ground state would again play an important role in implementing the concept of the minimal time-energy uncertainty relation.

For this purpose, we consider a hadron consisting of two quarks bound together by a harmonic oscillator force of the unit strength, and consider the following Lorentz-invariant differential equation. [6]

$$\frac{1}{2}\left[\frac{\partial^2}{\partial x_\mu{}^2}-x_\mu{}^2\right]\psi(x)=\lambda\psi(x) \qquad (1)$$

where x_μ denotes the space-time separation of the two quarks. The form of Eq. (1) has been extensively discussed in the literature. [7, 8, 9, 10] It has also been noted that the above differential equation can be separated in the following coordinate variables. [7]

$$\begin{aligned} x'&=x,\ y'=y \\ z'&=(z-\beta t)/(1-\beta^2)^{1/2} \\ t'&=(t-\beta z)/(1-\beta^2)^{1/2} \end{aligned} \qquad (2)$$

where β is the velocity paprameter of the hadron moving in the z direction. If we separate Eq. (1) in the above coordinate variables, we end up with four one-dimensional differential equations including the oscillator equation in the t' variable. This t' equation represents the existence of the time-energy uncertainty relation in the hadronic rest frame. If we impose the subsidiary condition

$$p^\mu\left(x_\mu+\frac{\partial}{\partial x^\mu}\right)\psi_\beta(x)=0 \qquad (3)$$

where p^μ is the four momentum of the hadron, then the solution in the t' variable is always in the ground state. This condition indeed makes the above-mentioned time-energy uncertainty minimal. [4]

The solutions of Eq. (1) satisfying the subsidiary condition of Eq. (3) are indeed compatible with the existing form of quantum mechanics and with the minimal time-energy uncertainty relation in the hadronic rest frame.

III. SPACE-TIME PICTURE OF HADRONIC DEFORMATION

Because the harmonic oscillator is separable, the transverse wave-functions do not play any essential roles in Lorentz transformations. We shall therefore ignore these transverse wave functions in the following discussion. Since the localization property of the harmonic oscillator wave function is dictated by its Gaussian factor, we shall restrict our discussion to the deformation property of the ground-state wave function.

If the hadron is at rest, the ground-state wave function takes the form

$$\psi_0(x) = \exp\left\{-\frac{1}{2}(z^2+t'^2)\right\} \qquad (4)$$

If the hadron moves with velocity β, then

$$\psi_\beta(x) = \exp\left\{-\frac{1}{2}(z'^2+t'^2)\right\} \qquad (5)$$

This expression represents that quantum mechanics remains unchanged if the observer moves with the hadron. If the observer chooses to stay in the original Lorentz frame, we have to express Eq. (5) in terms of z and t variables. After a simple algebra, Eq. (5) becomes

$$\psi_\beta(x) = \exp\left\{-\frac{1}{2}\left[\frac{1-\beta}{1+\beta}\left(\frac{z+t}{\sqrt{2}}\right)^2 + \frac{1+\beta}{1-\beta}\left(\frac{z-t}{\sqrt{2}}\right)^2\right]\right\} \qquad (6)$$

If $\beta=0$, this expression becomes Eq. (4).

Fig. 1 illustrates the space-time localization properties of the above hadronic wave functions. If $\beta=0$, the wave function is localized within the circular region:

$$(x^2+t^2)<2$$

If $\beta \neq 0$, then the wave function is localized within the elliptic region defined by

$$(x'^2+t'^2)<2. \qquad (8)$$

This condition can of course be rewritten as

$$\frac{1-\beta}{1+\beta}\left(\frac{z+t}{\sqrt{2}}\right)^2 + \frac{1+\beta}{1-\beta}\left(\frac{z-t}{\sqrt{2}}\right)^2 < 2. \qquad (9)$$

This elliptic region is also illustrated in Fig. 1. In Fig. 1, the lightcone axes serve as the major and minor axes for the ellipse. The area of ellipse is the same as that of the circle and remains invariant as β becomes large. This corresponds to the probability conservation which we write as

$$\int |\psi(x)|^2 d^4x = \int |\psi(x)|^2 d^4x \qquad (10)$$

If $\beta \to 1$, the wave function becomes concentrated along one of the light-cone axes. This is another way to understand the concentration of the interaction region along the light-cone axis which has been extensively discussed in the literature.[11, 12]

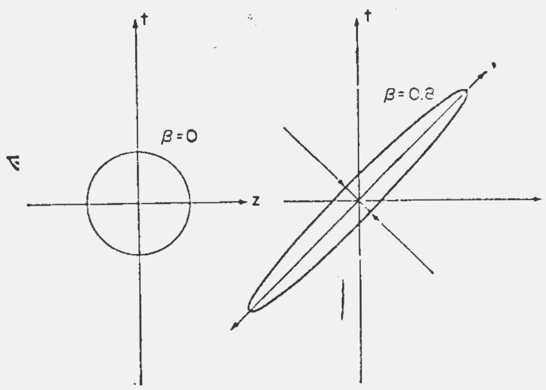

Fig. 1. Space-time picture of the Lorentz-deformed hadronic wave function in the harmonic oscillator quark model. As the hadron moves fast, the wave function becomes concentrated and elongated along one of the light-cone axes.

IV. PARTON PICTURE AS A MANIFESTATION OF THE MINIMAL TIME-ENERGY UNCERTAINTY RELATION

If we choose to learn lessons from history, space-time diagrams can sometimes play decisive roles in understanding the physics. The Feynman diagram is of course the prime example. In Sec. III, we used a pictorial language to study Lorentz-deformed hadrons in the harmonic-oscillator quark model. In this section, we add the hadronic deformation in the momentum space and solve the mystery of Feynman's parton phenomenon.

For this purpose, let us write the momentum wave function by taking the Fourier transformation of Eq. (5). or Eq. (6). The momentum wave function then becomes

$$\phi_\beta(q) = \exp\left\{-\frac{1}{2}\left[\frac{1-\beta}{1+\beta}\left(\frac{q_z+q_0}{\sqrt{2}}\right)^2 + \frac{1+\beta}{1-\beta}\left(\frac{q_z-q_0}{\sqrt{2}}\right)^2\right]\right\} \qquad (11)$$

where q represents the relative four momentum between the two bound-state quarks. This form of momentum wave function is sketched in Fig. 2 along with the space-time wave function of Eq. (6).

As $\beta \rightarrow 1$, both the space-time and momentum wave functions become concentrated and elongated along their respective light-cone axes. If we project these light-cone behaviors into the longitudinal axes, the mo mentum distribution becomes wider as the width of the spatial wave-function becomes larger. This unexpected peculiar behavior is called the parton phenomenon [5, 13] in high-energy physics.

The above mentioned behavior suggests that the uncertainty becomes larger for moving hadrons:

$$\langle \triangle z \rangle \langle \triangle p_z \rangle \simeq \frac{\hbar}{2}\left(\frac{1+\beta}{1-\beta}\right) \qquad (12)$$

This forces us to question whether Planck's

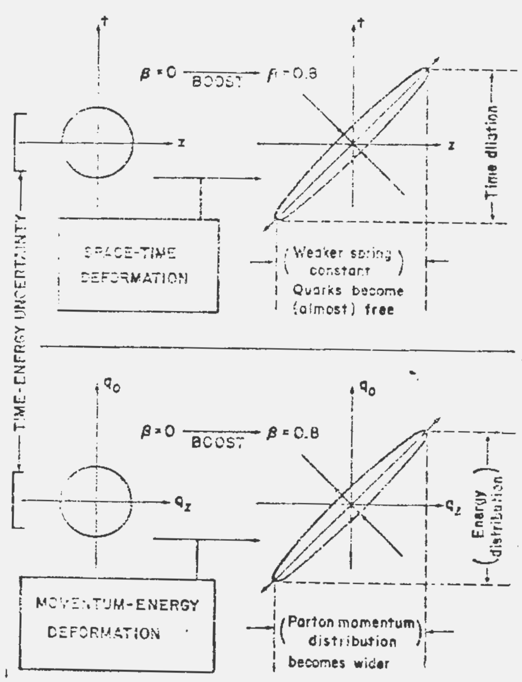

Fig. 2. The mechanism in which the hadronic deformation produces the peculiarities of Feynman's parton picture. Both the spatial and momentum wave functions become wide spread along the longitudinal axis, as the hadron moves fast.

constant is a Lorentz invariant concept.

In order to answer this question, we should note first that both the space-time and momentum wave functions are separable and Gaussian in the light-cone variables. In Fig. 2, the major axis of the space-time ellipse is conjugate to the minor axis of the momentum-energy ellipse. For this reason, the uncertainties defined along the light-cone axes remain Lorentz invariant. [13, 14] Planck's constant is indeed a Lorentz-invariant constant.

CONCLUDING REMARKS

In this paper, it was noted first that we need a time-energy uncertainty relation in order to make this world covariant. It was noted also that this time-energy uncertainty has to be minimal in order to be compatible with nature. We then used a diagramatic language to show that this minimal time-energy uncertainty relation, together with the longitudinal uncertainty, leads to the peculiarities in Feynman's parton Picture.

REFERENCES AND FOOTNOTES

[1] See articles by E.P Wigner, A. Shimony, J. S. Bell, and by D. Bohm, in *Foundations of Quantum Mechanics*, edited by B. D'Espagnat (Academic Press, New York and London, 1971).

[2] See articles by R. P. Feynman, J. Schwinger, and by S. Tomonaga, in *Quantum Electrodynamics*, edited by J. Schwinger (Dover, New York, 1958).

[3] The Lamb shift caculation is one of the successes of quantum electrodynamics. However, the covariant procedure of QED alone cannot produce the Rydberg formula to which the Lamb shift is a small correction. The Rydberg formula is still obtained from wave

functions satisfying the bound-state boundary condition.

[4] E.P. Wigner, in *Aspects of Quantum Theory, in Honour of P. A. M. Dirac's 70th Birthday*, edited by A. Salam and E. P. Wigner (Cambridge Univ. Press, London, 1972).

[5] R. P. Feynman, in *High Energy Collisions*, proceedings of the Third International Conference, Stony Brook, edited by C. N. Yang *et al.* (Gordon and Breach, New York, 1969).

[6] H. Yukawa, Phys. Rev. **91**, 416 (1953).

[7] Y. S. Kim and M. E. Noz, Phys. Rev. **D8**, 3521 (1973); **D12**, 129 (1975).

[8] Y. S. Kim, New Physics (Korean Phys. Soc.) **15**, 46 (1975); Phys. Rev. **D12**, 273 (1976).

[9] Y. S. Kim and M. E. Noz, Amer. J. Phys. (to be published).

[10] Y. S. Kim and M. E. Noz, Progr. Theor. Phys. **57**, 1373 (1977).

[11] B. L. Ioffe, Phys. Lett. **30B**, 123 (1969).

[12] For a comprehensive review of the light-cone physics, see Y. Frishman, in Proceedings of the Sixteenth International Conference on High Energy Physics, Vol. IV, edited by J. D. Jackson *et al.* (FNAL, 1972).

[13] Y. S. Kim and M. E. Noz, Phys. Rev. **D15**, 335 (1977).

[14] Y. S. Kim, J. Korean Phys. Soc. **9**, 54 (1976).