Photons in the Quantum World

Sibel Başkal
Department of Physics, Middle East Technical University, 06800 Ankara, Turkey

Young S. Kim
Center for Fundamental Physics, University of Maryland College Park, Maryland, MD 20742, USA

Marilyn E. Noz
Department of Radiology, New York University, New York, NY 10016, USA

Abstract

In the quantum world, the photon can have an angular momentum 1 or -1 along the direction of the momentum. In the Maxwell picture, the electromagnetic potentials become a four-vector, while the electric and magnetic fields are grouped into the four-by-four Maxwell tensor. In the quantum world, we start with spin-1/2 particles, with two spin states. In the Lorentz-covariant world, there are two additional spin states. There are thus four spin states for each spin-half particle, as in the case of the Dirac equation. The Cartesian product of two spinors leads to sixteen states, which can be partitioned into one for the scalar, one for the pseudo-scalar, four for the four-vector, four for the axial vector, and six for the tensor. This partition is worked out for a massive particle first. They can be Lorentz-boosted in the world of four spinors. It is then possible to take the infinite-momentum limit for the massless particle. This limit leads to a gauge-dependent four-vector for the four-potential and a gauge-independent tensor for the electromagnetic fields.

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1 Introduction

The algebra of quantum mechanics starts from Heisenberg's commutation relations

$$[x_i, p_j] = i\delta_{ij}, \tag{1}$$

with

$$p_i = -i\frac{\partial}{\partial x_i}. \tag{2}$$

These expressions are well known.

Let us next consider the operators

$$J_i = -i\epsilon_{ijk} x_j \frac{\partial}{\partial x_k} = -i\left(x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}\right), \tag{3}$$

which satisfy the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k. \tag{4}$$

This closed set of commutation relations is known as the Lie algebra of the three-dimensional rotation group or $O(3)$. This Lie algebra is a direct consequence of Heisenberg's uncertainty relations given in Eq.(1).

The simplest matrices representing this Lie algebra are

$$S_i = \frac{1}{2}\,\sigma_i, \tag{5}$$

where σ_i are the two-by-two Pauli spin matrices. We use the spinors

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{6}$$

for the spin-up and spin-down states respectively.

With these spinors, we can construct the spin-0 and spin-1 states in the following manner. For the spin-0 state, we make the anti-symmetric combination

$$\frac{1}{\sqrt{2}}(uv - vu). \tag{7}$$

There are three spin-1 states. They are

$$uu, \quad \frac{1}{\sqrt{2}}(uv + vu), \quad vv, \tag{8}$$

for the z-component spin 1, 0, and -1 respectively.

Next, we all know photons are massless particles with its spin 1 parallel or anti-parallel to its momentum. They are called helicities. We are also familiar with the expressions

$$A_\mu = \begin{pmatrix} A_0 \\ A_z \\ A_x \\ A_y \end{pmatrix}, \quad \text{and} \quad F_{\mu\nu} = \begin{pmatrix} 0 & -E_z & -E_x & -E_y \\ E_z & 0 & -B_y & B_x \\ E_x & B_y & 0 & -B_z \\ E_y & -B_x & B_z & 0 \end{pmatrix}. \tag{9}$$

These are Maxwell's four-vector and second-rank four-tensor for electromagnetic fields. These expressions belong to Einstein's Lorentz-covariant world. For convenience, we use the Minkowskian four-vector convention of (t, z, x, y) throughout the paper.

Table 1: Further contents of Einstein's $E = mc^2$. Under the Lorentz boost along the z direction, the z component of the spin remains as the helicity, but the transverse components collapse into one gauge degree of freedom.

	Slow	Relativistic	Fast
Energy-momentum	$p^2/2m$	$\sqrt{p^2 + m^2}$	$E = p$
Helicity Spin & Gauge	S_3 S_1, S_2	Wigner's Little Groups	Helicity Gauge Trans.

Here is the question. Is it possible to derive these Maxwell vector and tensor from Heisenberg's relations given in Eq.(1)? The answer is No, but is it possible to construct a bridge between them? The answer is Yes, but this question has a stormy history. The purpose of this paper is to provide a simple answer to this question. The bridge consists of the set of three two-by-two "imaginary" Pauli matrices $i\,\sigma_i$:

$$K_i = \frac{i}{2}\,\sigma_i. \tag{10}$$

They correspond to

$$K_i = -i\left(t\frac{\partial}{\partial x_i} + x_i\frac{\partial}{\partial t}\right), \tag{11}$$

applicable to the four-dimensional Minkowskian space.

The six matrices consisting of S_i and K_i become the generators of the group $SL(2,c)$ isomorphic to the group of Lorentz transformations. This group thus allows us to Lorentz-transform spinors which will eventually lead us to the electromagnetic four-vector and four-tensor.

It was of course Einstein who unified the energy-momentum relation for both massive and massless particles. We are now led to the problem of unifying internal space-time symmetries. Einstein's photon has spin-one parallel or anti-parallel to its momentum. For a particle at rest, we all know how to construct spin-1 states from two spinors as was the case in Eq.(8). The issue is how to Lorentz-boost those spin-1 states to reach the Maxwell tensor and four-vector for electromagnetic field.

It was Eugene Wigner who pioneered this research line. In 1939 [1], he constructed the subgroups of the Lorentz group whose transformations leave the four-momentum of a given particle invariant. He called these subgroups "little groups." Thus, Wigner's little groups dictate the internal space-time symmetries of particles in the Einstein's Lorentz-covariant world which includes both massive and massless particles, as shown in Table 1. This table was first published in 1986 [2]. Indeed, the photon polarization and the gauge degree of freedom are the issues of the internal space-time symmetries of massless particles.

In Sec. 2, we present two-by-two representation of the Lorentz group, and Wigner's little groups in Sec. 3. In Sec. 4, we discuss massless particles as large-momentum or small-mass limit of massive particles. It is shown that there are four spin states in the Lorentz-covariant world. Thus there are sixteen different ways to combine two spinors. In Sec. 5, we construct explicitly those sixteen states. Among them are the electromagnetic four-vector and the Maxwell tensor. It is shown that the polarization of massless neutrinos is a consequence of gauge invariance.

2 Lorentz Group and Its Representations

In addition to the rotation generators of Eq.(3), we can consider another set of three operators, namely

$$K_i = -i\left(x_i \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_i}\right). \tag{12}$$

These operators are known to generate Lorentz boosts in the Minkowskian space of one time direction and three spatial dimensions, and they satisfy the commutation relations

$$[K_i, K_j] = -i\epsilon_{ijk} J_k. \tag{13}$$

These three boost generators do not lead to a closed set of commutation relations. However, with the J_i generators, they satisfy the commutation relations

$$[J_i, K_j] = i\epsilon_{ijk} K_j. \tag{14}$$

Let us write the commutation relations of Eqs.(4), (13) and (14) as one closed set of commutation relations as

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} K_k. \tag{15}$$

This set is called the Lie algebra of the Lorentz group.

In terms of four-by-four matrices applicable to the Minkowskian coordinate of (t, z, x, y), the generators can be written as:

$$J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \qquad K_3 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{16}$$

for rotations around and boosts along the z direction, respectively. Similar expressions can be written for the x and y directions. We see here that the rotation generators J_i are Hermitian, but the boost generators K_i are anti-Hermitian.

The group of four-by-four matrices, which performs Lorentz transformations on the four-dimensional Minkowski space leaving invariant the quantity $(t^2 - z^2 - x^2 - y^2)$, forms the starting point for the Lorentz group. As there are three rotation and three boost generators, the Lorentz group is a six-parameter group.

The Lorentz group can also be represented by two-by-two matrices. If we choose

$$J_i = \frac{1}{2}\sigma_i, \quad K_i = \frac{i}{2}\sigma_i, \tag{17}$$

They satisfy the set commutation relations given in Eq.(15). Thus, to each two-by-two transformation matrix, there is a corresponding four-by-four matrix applicable to the Minkowskian space.

Table 2: Two-by-two representations of the Lorentz group. Rotations take the same form for both dotted and undotted representations, but boosts are performed in opposite directions.

Generators	Transformation Matrix		Dotted
$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} \exp(-i\phi/2) & 0 \\ 0 & \exp(i\phi/2) \end{pmatrix}$	same	$\begin{pmatrix} \exp(-i\phi/2) & 0 \\ 0 & \exp(i\phi/2) \end{pmatrix}$
$K_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} \exp(\eta/2) & 0 \\ 0 & \exp(-\eta/2) \end{pmatrix}$	inverse	$\begin{pmatrix} \exp(-\eta/2) & 0 \\ 0 & \exp(\eta/2) \end{pmatrix}$
$J_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$	same	$\begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$
$K_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$	inverse	$\begin{pmatrix} \cosh(\lambda/2) & -\sinh(\lambda/2) \\ -\sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$
$J_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$	same	$\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2 \end{pmatrix}$
$K_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} \cosh(\lambda/2) & -i\sinh(\lambda/2) \\ i\sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$	inverse	$\begin{pmatrix} \cosh(\lambda/2) & i\sinh(\lambda/2) \\ -i\sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$

The algebra of Eq.(17) is invariant under the sign change of the K_i matrices. Let us introduce the notation

$$\dot{K}_i = -K_i. \tag{18}$$

Then

$$J_i = \frac{1}{2}\sigma_i, \qquad \dot{K}_i = -\frac{i}{2}\sigma_i. \tag{19}$$

Corresponding to these two-by-two matrices, we can construct one set of two-component spinor (spin-up and spin-down) for the undotted representation, and another set for the dotted representation. There are thus four spin states in the Lorentz-covariant world as shown in Table 3.

As far as rotations are concerned, the representation constructed from the Lie algebra of Eq.(19) is transformed in the same way as that of Eq.(17). However, the Lorentz boosts are performed in opposite directions.

If two spinors are coupled, there are 16 (= 4 x 4) states, which can be partitioned into to the spin-0 and spin-1 states. We shall come back to this problem in Sec. 5.

Table 3: Spinors in the Relativistic world. The spinors u and v are for spin-up and spin-down states respectively. Under the Lorentz boost, the dotted spinors are boosted in the opposite direction.

	undotted	dotted
Spin up	u	$\dot{u}$
Spin down	v	$\dot{v}$

3 Wigner's Little Groups

There are interesting three-parameter subgroups of the Lorentz group. In 1939 [1], Wigner considered the subgroups whose transformations leave the four-momentum of a given particle invariant. First of all, consider a massive particle at rest. The momentum of this particle is invariant under rotations in three-dimensional space. What happens for the massless particle that cannot be brought to a rest frame? In this paper, we shall study this problem for understanding the space-time symmetry of photons.

The six generators of Eq.(15) lead to the group of two-by-two unimodular matrices of the form

$$G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{20}$$

with $\det(G) = 1$, where the matrix elements are complex numbers. There are thus six independent real numbers to accommodate the six generators given in Eq.(17). The groups of matrices of this form are called $SL(2,c)$ in the literature. Since the generators K_i are not Hermitian, the matrix G is not always unitary. Its Hermitian conjugate is not necessarily the inverse. This two-by-two representation has a rich history and has been discussed extensively in the literature [3, 4, 5, 6, 7, 8].

In this two-by-two representation, the space-time four-vector can be written as

$$\begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}, \tag{21}$$

whose determinant is $t^2 - z^2 - x^2 - z^2$, and remains invariant under the Hermitian transformation:

$$X' = G\ X\ G^{\dagger}. \tag{22}$$

This is thus a Lorentz transformation. This transformation can be explicitly written as:

$$\begin{pmatrix} t'+z' & x'-iy' \\ x'+iy' & t'-z' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}. \tag{23}$$

With these six independent real parameters, it is possible to construct four-by-four matrices for Lorentz transformations applicable to the four-dimensional Minkowskian space [7, 9]. For the purpose of the present paper, we need some special cases, and they are given in Table 2.

Likewise, the two-by-two matrix for the four-momentum takes the form

$$P = \begin{pmatrix} p_0 + p_z & p_x - ip_y \\ p_x + ip_y & p_0 - p_z \end{pmatrix}, \tag{24}$$

with $p_0 = \sqrt{m^2 + p_z^2 + p_x^2 + p_2^2}$. The transformation property of Eq.(23) is applicable also to this energy-momentum four-vector. In 1939 [1], Wigner considered the following three four-vectors.

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{25}$$

whose determinants are 1, 0, and -1, respectively, corresponding to the four-momenta of massive, massless, and imaginary-mass particles, as shown in Table 4.

He then constructed the subgroups of the Lorentz group whose transformations leave these four-momenta invariant. These subgroups are called Wigner's little groups in the literature. Thus, the matrices of these little groups should satisfy:

$$W \ P_i \ W^\dagger = P_i, \tag{26}$$

where $i = +, 0, -$. Since the momentum of the particle is fixed, these little groups define the internal space-time symmetries of the particle.

For all three cases, the momentum is invariant under rotations around the z axis. The matrix for these rotations is given in Table 4. Let us write this rotation matrix

$$Z(\phi) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}. \tag{27}$$

Then

$$Z(\phi) \ P_i \ Z^\dagger(\phi) = P_i. \tag{28}$$

This means that the four-momentum remains invariant under rotations around the z axis.

In addition, let us consider the transformation within the zx plane. For the first case corresponding to a massive particle at rest, the requirement of the subgroup is:

$$W \ P_+ \ W^\dagger = P_+. \tag{29}$$

This four-momentum remains invariant under rotations around the y axis, whose transformation matrix is

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \tag{30}$$

This matrix together with $Z(\phi)$ leads to the rotation also around the x axis. Thus, Wigner's little group for the massive particle is the three-dimensional rotation subgroup of the Lorentz group generated by S_i given in Eq.(5).

For the second case of P_0, the triangular matrix of the form:

$$T(\gamma) = \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}, \quad \text{or} \quad \dot{T}(\gamma) = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \tag{31}$$

satisfies the Wigner condition of Eq.(26). If we allow rotations around the z axis, these triangular matrices become

$$T\left(\gamma e^{-i\phi}\right) = \begin{pmatrix} 1 & -\gamma \exp(-i\phi) \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \dot{T}\left(\gamma e^{-i\phi}\right) = \begin{pmatrix} 1 & 0 \\ \gamma \exp(i\phi) & 1 \end{pmatrix}. \tag{32}$$

The T matrix is generated by:

$$N_1 = J_2 - K_1 = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad N_2 = J_1 + K_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{33}$$

Table 4: The two-by-two matrix representation of Wigner momentum vectors together with the corresponding transformation matrix. These four-momentum matrices have determinants that are positive, zero, and negative for massive, massless, and imaginary-mass particles, respectively.

Mass	4-momentum	Transformation Matrix	Dotted
Massive	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$	$\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$
Massless	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$
Imag. mass	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$	$\begin{pmatrix} \cosh(\lambda/2) & -\sinh(\lambda/2) \\ -\sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$

Its dotted matrix is generated by

$$\dot{N}_1 = J_2 + K_1 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \dot{N}_2 = J_1 - K_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{34}$$

Thus, the little group is generated by J_3, N_1, and N_2. Together with J_3 they satisfy the following sets of commutation relations:

$$[N_1, N_2] = 0, \qquad [J_3, N_1] = iN_2, \qquad [J_3, N_2] = -iN_1. \tag{35}$$

and

$$\left[\dot{N}_1, \dot{N}_2\right] = 0, \qquad \left[J_3, \dot{N}_1\right] = i\dot{N}_2, \qquad \left[J_3, \dot{N}_2\right] = -i\dot{N}_1. \tag{36}$$

Wigner in 1939 [1] observed that the first set given in Eq.(35) is the same as that of the generators for the two-dimensional Euclidean group with one rotation and two translations. The physical interpretation of the rotation is easy to understand. It is the helicity of the massless particle. On the other hand, the physics of the N_1 and N_2 matrices has a stormy history, and the issue was not completely settled until 1990 [10]. They generate gauge transformations [14, 15, 16].

For the third case of P_-, the matrix of the form:

$$S(\lambda) = \begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}, \tag{37}$$

satisfies the Wigner condition of Eq.(26). This corresponds to the Lorentz boost along the x direction generated by K_1 as shown in Table 2. Because of the rotational symmetry around the z axis, the Wigner condition is satisfied also by the boost along the y axis. The little group is thus generated by J_3, K_1, and K_2. These three generators:

$$[J_3, K_1] = iK_2, \qquad [J_3, K_2] = -iK_1, \qquad [K_1, K_2] = -iJ_3 \tag{38}$$

Table 5: $T(\gamma)$ and $\dot{T}(\gamma)$ transformations on the spinors. Due to the parity invariance of the Lie algebra of the Lorentz group, we should consider the triangular matrices and their dots applicable to both u and v, and and also to $\dot{u}$ and $\dot{v}$.

	$T(\gamma)$ with $+\eta$	$\dot{T}(\gamma)$ with $-\eta$
Spinors	$T(\gamma)u = u$	$\dot{T}(\gamma)u = u + \gamma v$
	$T(\gamma)v = v - \gamma u$	$\dot{T}(\gamma)v = v$
Dotted spinors	$T(\gamma)\dot{u} = \dot{u}$	$\dot{T}(\gamma)\dot{u} = \dot{u} + \gamma\dot{v}$
	$T(\gamma)\dot{v} = \dot{v} - \gamma\dot{u}$	$\dot{T}(\gamma)\dot{v} = \dot{v}$

form the little group $O(2,1)$, which is the Lorentz group applicable to two space-like and one time-like dimensions. The dotted matrices should satisfy the same set of commutation relations.

4 Massive and Massless Particles

Indeed, the massive particle at rest remains invariant under rotations. Let us Lorentz-boost this particle along the z direction. The boost matrix is given in Table 2, and it takes the form

$$B(\eta) = \begin{pmatrix} \exp(\eta/2) & 0 \\ 0 & \exp(-\eta/2) \end{pmatrix}. \tag{39}$$

Then its momentum becomes

$$p_z = m\ \sinh(\eta), \quad \text{or} \quad e^{\eta} = \frac{p_z + \sqrt{p_z^2 + m^2}}{m}, \tag{40}$$

This momentum remains invariant under rotations around the axis. The rotation matrix $Z(\phi)$ given in Eq.(27) commutes with the boost matrix $B(\eta)$ of Eq.(39).

The story is different for rotations around an axis perpendicular to the z axis. Let us pick the rotation around the y axis given in Eq.(30). This matrix becomes boosted to

$$B(\eta)R(\theta)B^{\dagger}(-\eta) = \begin{pmatrix} \cos(\theta/2) & -e^{\eta}\sin(\theta/2) \\ e^{-\eta}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \tag{41}$$

where the boost matrix $B(\eta)$ of Eq.(39. According to Eq.(40), η becomes infinite as the mass becomes smaller. If we decide to keep all the quantities in Eq.(41) finite, the upper-right element $e^{\eta}\sin(\theta/2)$ must be finite. Let that be γ. The lower-left element then becomes $e^{-2\eta}\gamma$ which vanishes as η becomes infinite. The angle θ becomes zero. Thus, the boosted rotation matrix becomes the triangular matrix

$$T = \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \dot{T} = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}, \tag{42}$$

which are the triangular Wigner matrices given in Eq.(31). When they are applied to the spinors given in Table 3, u and $\dot{v}$ remain invariant, but $\dot{u}$ and v become changed as shown in Table 5.

Here again, there is the rotational degree of freedom around the z axis. The matrix of Eq.(41) is generalized into

$$\begin{pmatrix} 0 & e^{-i\phi/2} \\ e^{i\phi/2} & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & -e^{\eta}\sin(\theta/2) \\ e^{-\eta}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} 0 & e^{i\phi/2} \\ e^{-i\phi/2} & 0 \end{pmatrix}, \tag{43}$$

which becomes

$$\begin{pmatrix} \cos(\theta/2) & -e^{-i\phi}e^{\eta}\sin(\theta/2) \\ e^{i\phi}e^{-\eta}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}. \tag{44}$$

In the large-η limit, this expression leads to the triangular matrices of Eq.(32).

5 Scalars, Vectors, and Tensors

We are quite familiar with the process of constructing three spin-1 states and one spin-0 state from two spinors. Since each spinor has two states, there are four states if combined.

In the Lorentz-covariant world, for each spin-1/2 particle, there are two additional two-component spinors coming from the dotted representation [7, 9, 11, 12, 13]. There are thus four states. If two spinors are combined, there are 16 states. They can be partitioned into the following states.

1. scalar with one state,
2. pseudo-scalar with one state,
3. four-vector with four states,
4. axial vector with four states,
5. second-rank tensor with six states.

If the particle is at rest, we can explicitly construct the combinations

$$uu, \qquad \frac{1}{\sqrt{2}}(uv + vu), \qquad vv, \tag{45}$$

to obtain the spin-1 states and

$$\frac{1}{\sqrt{2}}(uv - vu), \tag{46}$$

for the spin-zero state. This results in four bilinear states. In the $SL(2,c)$ regime, there are two dotted spinors. If we include both dotted and undotted spinors, there are sixteen independent bilinear combinations. They are given in Table 6.

Among the bilinear combinations given in Table 6, the following two equations are invariant under rotations and also under boosts:

$$S = \frac{1}{\sqrt{2}}(uv - vu), \quad \text{and} \quad \dot{S} = -\frac{1}{\sqrt{2}}(\dot{u}\dot{v} - \dot{v}\dot{u}). \tag{47}$$

These are thus scalars in the Lorentz-covariant world. Are they the same or different? Let us consider the following combinations

$$S_{+} = \frac{1}{\sqrt{2}}\left(S + \dot{S}\right), \quad \text{and} \quad S_{-} = \frac{1}{\sqrt{2}}\left(S - \dot{S}\right). \tag{48}$$

Table 6: Sixteen combinations of the $SL(2,c)$ spinors. In the $SU(2)$ regime, there are two spinors leading to four bilinear forms. In the $SL(2,c)$ world, there are two undotted and two dotted spinors. These four-spinors lead to sixteen independent bilinear combinations.

Spin 1			Spin 0
uu,	$\frac{1}{\sqrt{2}}(uv+vu)$,	vv,	$\frac{1}{\sqrt{2}}(uv-vu)$
$\dot{u}\dot{u}$,	$\frac{1}{\sqrt{2}}(\dot{u}\dot{v}+\dot{v}\dot{u})$,	$\dot{v}\dot{v}$,	$\frac{1}{\sqrt{2}}(\dot{u}\dot{v}-\dot{v}\dot{u})$
$u\dot{u}$,	$\frac{1}{\sqrt{2}}(u\dot{v}+v\dot{u})$,	$v\dot{v}$,	$\frac{1}{\sqrt{2}}(u\dot{v}-v\dot{u})$
$\dot{u}u$,	$\frac{1}{\sqrt{2}}(\dot{u}v+\dot{v}u)$,	$\dot{v}v$,	$\frac{1}{\sqrt{2}}(\dot{u}v-\dot{v}u)$

Under the dot conjugation, S_+ remains invariant, but S_- changes sign. As was noted in Sec. 2, the dot conjugation corresponds to space inversion. Thus, S_+ is a scalar, while S_- is called a pseudo-scalar.

5.1 Four-vectors

Let us rewrite the expression for the space-time four-vector given in Eq.(21) as

$$\begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}, \tag{49}$$

which, under the parity operation, becomes

$$\begin{pmatrix} t-z & -x+iy \\ -x-iy & t+z \end{pmatrix}. \tag{50}$$

The off-diagonal elements undergo sign changes, and the diagonal elements become interchanged.

We can now construct the four-vectors like Eq.(49) and its "dot" conjugation as

$$V \simeq \begin{pmatrix} u\dot{v}-\dot{v}u & v\dot{v}-\dot{v}v \\ u\dot{u}-\dot{u}u & \dot{u}v-v\dot{u} \end{pmatrix}, \qquad \dot{V} \simeq \begin{pmatrix} \dot{u}v-v\dot{u} & \dot{v}v-v\dot{v} \\ \dot{u}u-u\dot{u} & u\dot{v}-\dot{v}u \end{pmatrix}, \tag{51}$$

respectively.

Accordingly, we write the electromagnetic four-potential as

$$A = \begin{pmatrix} A_0+A_z & A_x-iA_y \\ A_x+iA_y & A_0-A_z \end{pmatrix}. \tag{52}$$

If boosted along the z direction, this matrix becomes

$$A = \begin{pmatrix} (A_0+A_z)\,e^{\eta} & A_x-iA_y \\ A_x+iA_y & (A_0-A_z)\,e^{-\eta} \end{pmatrix}. \tag{53}$$

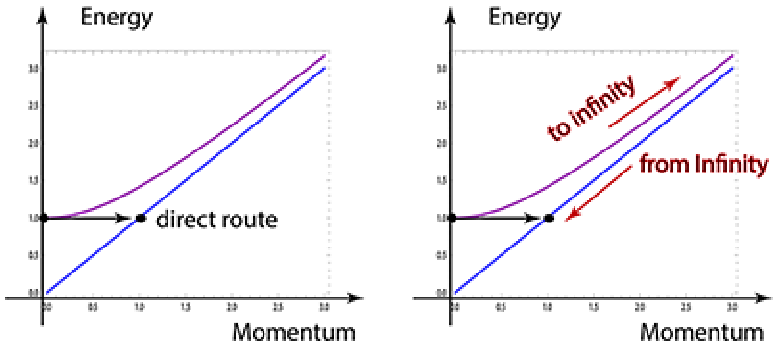

Figure 1: Wigner excursion. If we are interested in converting a massive particle at rest, we can to boost it to an infinite-momentum along the hyperbola in this figure, where it concides with the light come. We can then come back along the light cone.

We can then make the Wigner excursion as illustrated in Fig. 1, which transforms this matrix for a massive particle at rest to that of a massless particle with the same energy. The net result is

$$A = \begin{pmatrix} A_0 + A_z & A_x - iA_y \\ A_x + iA_y & 0 \end{pmatrix}, \tag{54}$$

resulting in $A_0 = A_z$ which is widely known as the Lorentz condition.

If we perform the $T(\gamma)$ on u and v, while $\dot{T}(\gamma)$ on $\dot{u}$ and $\dot{v}$, the A matrix becomes

$$A + 2\gamma \begin{pmatrix} 0 & A_0 \\ A_0 & 0 \end{pmatrix}, \tag{55}$$

This results in the addition of $2\gamma A_0$ to A_x. It is a translation in the plane of A_x and A_y.

On the other hand, if we perform the $\dot{T}(\gamma)$ on u and v, while $T(\gamma)$ on $\dot{u}$ and $\dot{v}$, A becomes

$$A + 2\gamma \begin{pmatrix} A_x & 0 \\ 0 & 0 \end{pmatrix}. \tag{56}$$

Those triangular matrices are in Table 5. This means that the $T(\gamma)$ performs a gauge transformation.

What we have done so far can be rotated around z axis. Then, γ is replaced by $\gamma e^{-i\phi}$. The transformed A of Eq.(55) becomes

$$A + 2\gamma A_0 \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}, \tag{57}$$

and the matrix of Eq.(56) becomes

$$A + 2\gamma \begin{pmatrix} A_x \cos\phi + A_y \sin\phi & 0 \\ 0 & 0 \end{pmatrix}. \tag{58}$$

It is possible to reach the same conclusion using the four-by-four formulation of the Lorentz group. This larger representation contains geometries leading to Eq.(57) and Eq.(58) [10, 14, 15, 16]. We now know from Eq.(58) that $T\left(\gamma e^{-i\phi}\right)$ performs a gauge transformation.

Let us go back to the limiting process discussed in Sec. 4. According to Sec. 4, the transverse rotational degrees of freedom collapse into one gauge degree of freedom in the infinite-momentum or zero-mass limit. This aspect was observed first by Han *et al.* in 1983 [17], and its geometry was given by Kim and Wigner in 1990 [10]. The most recent version of this geometry was given by the present authors in 2017 [8].

5.2 Second-Rank Tensor

There are also bilinear spinors, which are both dotted or both undotted. We are interested in two sets of three quantities satisfying the $O(3)$ symmetry. They should therefore transform like:

$$(x+iy)/\sqrt{2}, \qquad (x-iy)/\sqrt{2}, \qquad z, \tag{59}$$

which are like:

$$uu, \qquad vv, \quad (uv+vu)/\sqrt{2}, \tag{60}$$

respectively, in the $O(3)$ regime. Since the dot conjugation is the parity operation, they are like:

$$-\dot{u}\dot{u}, \qquad -\dot{v}\dot{v}, \qquad -(\dot{u}\dot{v}+\dot{v}\dot{u})/\sqrt{2}. \tag{61}$$

In other words,

$$\dot{(uu)} = -\dot{u}\dot{u}, \quad \text{and} \quad \dot{(vv)} = -\dot{v}\dot{v}. \tag{62}$$

We noticed a similar sign change in Eq,(50).

In order to construct the z component in this $O(3)$ space, let us first consider:

$$f_z = \frac{1}{2}\left[(uv+vu) - (\dot{u}\dot{v}+\dot{v}\dot{u})\right], \qquad g_z = \frac{1}{2i}\left[(uv+vu) + (\dot{u}\dot{v}+\dot{v}\dot{u})\right]. \tag{63}$$

Here, f_z and g_z are respectively symmetric and anti-symmetric under the dot conjugation or the parity operation. These quantities are invariant under the boost along the z direction. They are also invariant under rotations around this axis, but they are not invariant under boosts along or rotations around the x or y axis. They are different from the scalars given in Eq.(47).

Next, in order to construct the x and y components, we start with $f_\pm$ and $g_\pm$ as:

$$\begin{aligned} f_+ &= \frac{1}{\sqrt{2}}\left(uu - \dot{u}\dot{u}\right), \qquad f_- = \frac{1}{\sqrt{2}}\left(vv - \dot{v}\dot{v}\right), \\ g_+ &= \frac{1}{\sqrt{2}i}\left(uu + \dot{u}\dot{u}\right), \qquad g_- = \frac{1}{\sqrt{2}i}\left(vv + \dot{v}\dot{v}\right). \end{aligned} \tag{64}$$

Then:

$$\begin{aligned} f_x &= \frac{1}{\sqrt{2}}\left(f_+ + f_-\right) = \frac{1}{2}\left[(uu+vv) - (\dot{u}\dot{u}+\dot{v}\dot{v})\right], \\ f_y &= \frac{1}{\sqrt{2}i}\left(f_+ - f_-\right) = \frac{1}{2i}\left[(uu-vv) - (\dot{u}\dot{u}-\dot{v}\dot{v})\right], \end{aligned} \tag{65}$$

and:

$$\begin{aligned} g_x &= \frac{1}{\sqrt{2}}\left(g_+ + g_-\right) = \frac{1}{2i}\left[(uu+vv) + (\dot{u}\dot{u}+\dot{v}\dot{v})\right], \\ g_y &= \frac{1}{\sqrt{2}i}\left(g_+ - g_-\right) = -\frac{1}{2}\left[(uu-vv) + (\dot{u}\dot{u}-\dot{v}\dot{v})\right]. \end{aligned} \tag{66}$$

Here, f_x and f_y are anti-symmetric under dot conjugation, while g_x and g_y are symmetric.

Furthermore, f_z, f_x and f_y of Eqs. (63) and (65) transform like a three-dimensional vector. The same can be said for g_i of Eqs. (63) and (66). Thus, they can be grouped to form a second-rank tensor:

$$\begin{pmatrix} 0 & -f_z & -f_x & -f_y \\ f_z & 0 & -g_y & g_x \\ f_x & g_y & 0 & -g_z \\ f_y & -g_x & g_z & 0 \end{pmatrix}, \tag{67}$$

whose Lorentz-transformation properties are well known. The g_i components change their signs under space inversion, while the f_i components remain invariant. They are like the electric and magnetic fields, respectively.

If the system is Lorentz-boosted, f_i and g_i can be computed from Table 6. We are now interested in the symmetry of photons by taking the massless limit. Thus, we keep only the terms that become larger for larger values of η. Thus,

$$f_x \to \frac{1}{2}(uu - \dot{v}\dot{v}), \qquad f_y \to \frac{1}{2i}(uu + \dot{v}\dot{v}),$$

$$g_x \to \frac{1}{2i}(uu + \dot{v}\dot{v}), \qquad g_y \to -\frac{1}{2}(uu - \dot{v}\dot{v}), \tag{68}$$

in the massless limit.

Then, the tensor of Eq.(67) becomes:

$$\begin{pmatrix} 0 & 0 & -E_x & -E_y \\ 0 & 0 & -B_y & B_x \\ E_x & B_y & 0 & 0 \\ E_y & -B_x & 0 & 0 \end{pmatrix}, \tag{69}$$

with:

$$E_x \simeq \frac{1}{2}(uu - \dot{v}\dot{v}), \qquad E_y \simeq \frac{1}{2i}(uu + \dot{v}\dot{v}),$$

$$B_x \simeq \frac{1}{2i}(uu + \dot{v}\dot{v}), \qquad B_y \simeq -\frac{1}{2}(uu - \dot{v}\dot{v}). \tag{70}$$

The four-by-four matrix of Eq.(69) is consistent with the Maxwell tensor given in Eq.(9).

The electric and magnetic field components are perpendicular to each other. Furthermore,

$$B_x = E_y, \qquad B_y = -E_x. \tag{71}$$

In order to address symmetry of photons, let us go back to Eq.(64). In the massless limit,

$$B_+ \simeq E_+ \simeq uu, \qquad B_- \simeq E_- \simeq \dot{v}\dot{v}. \tag{72}$$

The gauge transformations applicable to u and $\dot{v}$ are the two-by-two matrices:

$$\begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}, \tag{73}$$

respectively. Both u and $\dot{v}$ are invariant under gauge transformations, while $\dot{u}$ and v are not.

The B_+ and E_+ are for the photon spin along the z direction, while B_- and E_- are for the opposite direction.

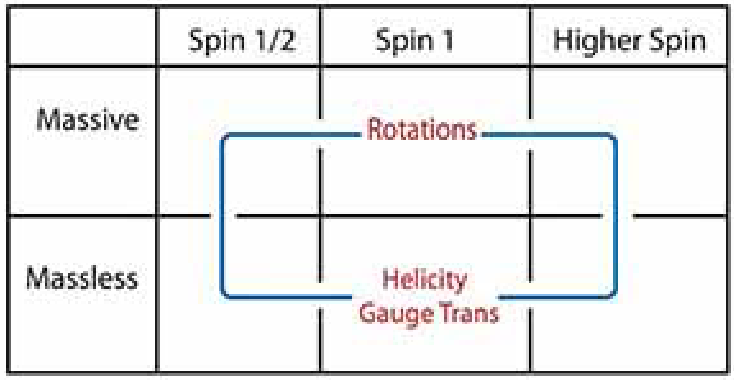

Figure 2: Unified picture of massive and massless particles. The gauge transformation is a Lorentz-boosted rotation matrix and is applicable to all massless particles. It is possible to construct higher-spin states starting from the four states of the spin-1/2 particle in the Lorentz-covariant world.

5.3 Higher Spins

Since Wigner's original book of 1931 [18, 19], the rotation group, without Lorentz transformations, has been extensively discussed in the literature [12, 20, 21]. One of the main issues was how to construct the most general spin state from the two-component spinors for the spin-1/2 particle.

Since there are two states for the spin-1/2 particle, four states can be constructed from two spinors, leading to one state for the spin-0 state and three spin-1 states. With three spinors, it is possible to construct four spin-3/2 states and two spin-1/2 states, resulting in six states. This partition process is much more complicated [22, 23] for the case of three spinors. Yet, this partition process is possible for all higher spin states.

In the Lorentz-covariant world, there are four states for each spin-1/2 particle. With two spinors, we end up with sixteen (4×4) states, and they are tabulated in Table 6. There should be 64 states for three spinors and 256 states for four spinors. We now know how to Lorentz-boost those spinors. We also know that the transverse rotations become gauge transformations in the limit of zero-mass or infinite-η. It is thus possible to bundle all of them into the table given in Figure 2.

In the relativistic regime, we are interested in photons and gravitons. As was noted in Subsections 5.1 and 5.2, the observable components are invariant under gauge transformations. They are also the terms that become largest for large values of η.

We have seen in Section 5.2 that the photon state consists of uu and $\dot{v}\dot{v}$ for those whose spins are parallel and anti-parallel to the momentum, respectively. Thus, for spin-2 gravitons, the states must be $uuuu$ and $\dot{v}\dot{v}\dot{v}\dot{v}$, respectively.

In his effort to understand photons and gravitons, Weinberg constructed his states for massless particles [24], especially photons and gravitons [25]. He started with the conditions:

$$N_1|\text{state}>= 0, \quad \text{and} \quad N_2|\text{state}>= 0, \tag{74}$$

where N_1 and N_2 are defined in Eq.(33). Since they are now known as the generators of gauge transformations, Weinberg's states are gauge-invariant states. Thus, uu and $\dot{v}\dot{v}$ are Weinberg's states for photons, and $uuuu$ are $\dot{v}\dot{v}\dot{v}\dot{v}$ are Weinberg's states for gravitons.

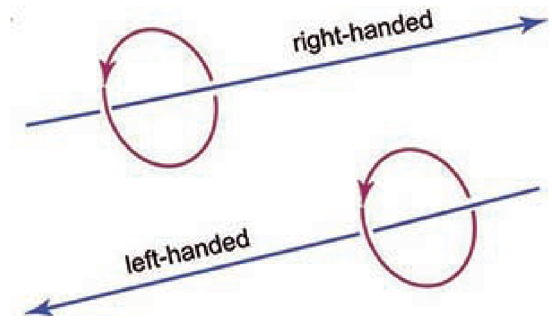

Figure 3: Polarization of massless neutrinos. This polarization is a consequence of gauge invariance.

5.4 Polarization of Massless Neutrinos

We have established the triangular matrices T and $\dot{T}$ generate gauge transformations when applied to four-vectors. Let us go back to Table 5. They also perform gauge transformations on massless spin-half particles [2, 26].

Let us go back to Table 3. If we insist on gauge invariance of the world, spin-half particles are polarized. The dotted particle becomes left-handed, while the undotted spinor becomes right-handed [2, 26, 27]. Indeed, this is what we observe in the real world. Massless neutrinos and anti-neutrinos are left- and right-handed respectively.

Yes, neutrinos have non-zero masses [28, 29], but they are so small compared with their momenta that they can be regarded as small corrections to their massless states. In other words, their massless states will play important roles in physics.

Concluding Remarks

It was Einstein who defined the photon as a massless particle in the quantum world from his photo-electric effect. However, he did not consider the "wings" of the electromagnetic wave. In the classical picture, there are electric and magnetic fields perpendicular to the direction of propagation. This aspect is translated into the polarization of photons.

This question belongs to the subject area pioneered by Wigner in 1939 [1]. His 1939 deals with the internal space-time symmetries, as specified in Table 1. However, the issue of the electromagnetic four-potential with its gauge degree of freedom has a stormy history and was settled in later papers [10, 14, 16]. As for the Maxwell tensor, the present authors dealt with the problem in their recent publications [7, 8]. In this report, we have presented further details of this problem starting from the transformation properties of the four spinors defined for the Lorentz-covariant world.

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