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Cylindrical Group and Massless Particles

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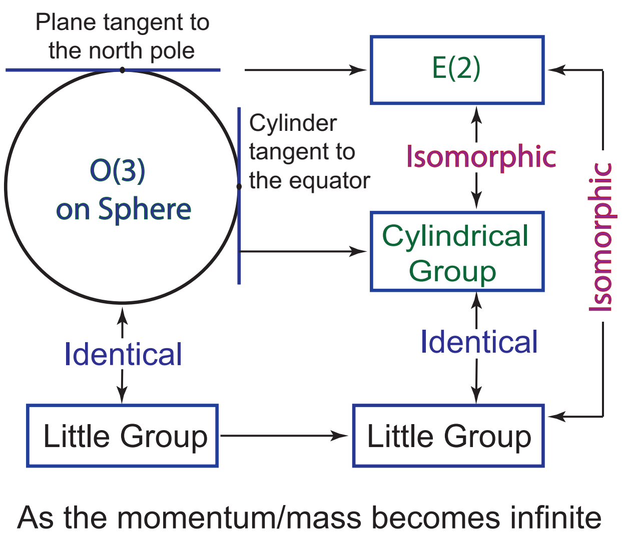
Abstract

It is shown that the representation of the $E(2)$-like little group for photons can be reduced to the coordinate transformation matrix of the cylindrical group, which describes movement of a point on a cylindrical surface. The cylindrical group is isomorphic to the two-dimensional Euclidean group. As in the case of $E(2)$, the cylindrical group can be regarded as a contraction of the three-dimensional rotation group. It is pointed out that the $E(2)$-like little group is the Lorentz-boosted $O(3)$-like little group for massive particles in the infinite-momentum/zero-mass limit. This limiting process is shown to be identical to that of the contraction of $O(3)$ to the cylindrical group. Gauge transformations for free massless particles can thus be regarded as Lorentz-boosted rotations.



1 Introduction

In their 1953 paper [1], Inonu and Wigner discussed the contraction of the three-dimensional rotation group [or $O(3)$] to the two-dimensional Euclidean group [or $E(2)$]. Since the little groups governing the internal space-time symmetries of massive and massless particles are locally isomorphic to $O(3)$ and $E(2)$ respectively [2], it is quite natural for us to expect that the $E(2)$-like little group is a limiting case of the $O(3)$-like little group [3].

The kinematics of the $O(3)$-like little group for a massive particle is well understood. The identification of this little group with $O(3)$ can best be achieved in the Lorentz frame in which the particle is at rest [2]. In this frame, we can rotate the direction of the spin without changing the momentum. Indeed, for a massive particle, the little group is for the description of the spin orientation in the rest frame.

The kinematics of the $E(2)$-like little group has been somewhat less transparent, because there is no Lorentz frame in which the particle is at rest. While the geometry of $E(2)$ can best be understood in terms of rotations and translations in two-dimensional space, there is no physical reason to expect that the translation-like degrees of freedom in the $E(2)$-like little group represent translations in an observable space. In fact, the translation-like degrees of freedom in the little group are the gauge degrees of freedom [3]. Therefore, in the past, the correspondence between the $E(2)$-like little group and the two-dimensional Euclidean group has been strictly algebraic.

In this paper, we formulate a group theory of a point moving on the surface of a circular cylinder. This group is locally isomorphic to the two-dimensional Euclidean group. We show that the transformation matrix of the little group for photons reduces to that of the coordinate transformation matrix of the cylindrical group. The cylindrical group therefore bridges the gap between $E(2)$ and the $E(2)$-like little group.

As in the case of $E(2)$, we can obtain the cylindrical group by contracting the three-dimensional rotation group. While the contraction of $O(3)$ to $E(2)$ is a tangent-plane approximation of a spherical surface with large radius [1], the contraction to the cylindrical group is a tangent-cylinder approximation. Using this result, together with the fact that the representation of the $E(2)$-like little group reduces to that of the cylindrical group, we show that the gauge degree of freedom for massless particles come from Lorentz-boosted rotations.

In Sec. 2, we discuss the cylindrical group and its isomorphism to the two-dimensional Euclidean group. Section 3 deals with the $E(2)$-like little group for photons and its isomorphism to the cylindrical group. It is shown in Sec. 4 that the cylindrical group can be regarded as an equatorial-belt approximation of the three-dimensional rotation group, while $E(2)$ can be regarded as a north-pole approximation. In Sec. 5, we combine the conclusions of Sec. 3 and Sec. 4 to show that the gauge degrees of freedom for free massless particles are Lorentz-boosted rotational degrees of freedom.

2 Two-dimensional Euclidean Grouip and Cylindrical group

The two-dimensional Euclidean group, often called $E(2)$, consists of rotations and translations on a two-dimensional Euclidian plane. The coordinate transformation takes the form

$$x' = x\cos\alpha - y\sin\alpha + u, \qquad y' = x\sin\alpha + y\cos\alpha + v. \tag{1}$$

This transformation can be written in matrix form as

$$\begin{pmatrix} u' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & u \\ \sin\alpha & \cos\alpha & v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ y \\ 1 \end{pmatrix} \tag{2}$$

The three-by-three matrix in the above expression can be exponentiated as

$$E(u, v, \alpha) = \exp\left(-i(uP_1 + vP_2)\right) \exp\left(-i\alpha L_3\right), \tag{3}$$

where L_3 is the generator of rotations, and P_1 and P_2 generate translations. These generators take the form

$$L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \tag{4}$$

and satisfy the commutation relations:

$$[P_1, P_2] = 0, \qquad [L_3, P_1] = iP_2, \qquad [L_3, P_2] = -iP_1, \tag{5}$$

which form the Lie algebra for $E(2)$.

The above commutation relations are invariant under the sign change in P_1 and P_2. They are also invariant under Hermitian conjugation. Since L_3 is Hermitian, we can replace P_1 and P_2 by

$$Q_1 = -P_1^{\dagger}, \qquad Q_2 = -P_2^{\dagger}, \tag{6}$$

respectively to obtain

$$[Q_1, Q_2] = 0, \qquad [L_3, Q_1] = iQ_2, \qquad [L_3, Q_2] = -iQ_1, \tag{7}$$

These above commutation relations are identical to those for $E(2)$ given in Eq.(5). However, Q_1 and Q_2 are not the generators of Euclidean translations in the two-dimensional space. Let us write down their matrix forms:

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \tag{8}$$

while L_3 is given in Eq.(4). As in the case of $E(2)$, we can consider the transformation matrix:

$$C(u, v, \alpha) = C(0, 0, \alpha)C(u, v, 0), \tag{9}$$

where $C(0, 0, \alpha)$ is the rotation matrix and takes the form

$$C(0, 0, \alpha) = \exp\left(-i\alpha L_3\right) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{10}$$

and

$$C(u, v, 0) = \exp\left[-i(uQ_1 + vQ_2)\right] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & v & 1 \end{pmatrix}. \tag{11}$$

The multiplication of the above two matrices results in the most general form of $C(u, v, \alpha)$. If this matrix is applied to the column vector (x, y, z), the result is

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ u & v & 1 \end{pmatrix} \begin{pmatrix} u \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos\alpha - y\sin\alpha \\ x\sin\alpha + y\cos\alpha \\ z + ux + vy \end{pmatrix}. \tag{12}$$

This transformation leaves $(x^2 + y^2)$ invariant, while z can vary from $-\infty$ to $+\infty$. For this reason, it is quite appropriate to call the group of the above linear transformation the cylindrical group. This group is locally isomorphic to $E(2)$.

If, for convenience, we set the radius of the cylinder to be unity:

$$x^2 + y^2 = 1, \tag{13}$$

then x and y can be written as

$$x = \cos\phi, \qquad y = \sin\phi, \tag{14}$$

and the transformation of Eq.(12) takes the form

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ u & v & 1 \end{pmatrix} \begin{pmatrix} \cos\phi \\ \sin\phi \\ z \end{pmatrix} = \begin{pmatrix} \cos(\phi + \alpha) \\ \sin(\phi + \alpha) \\ z + (u\cos\phi + v\sin\phi) \end{pmatrix}. \tag{15}$$

We shall see in the following sections how this cylindrical group describes gauge transformations for massless particles.

3 E(2)-like Little Group for Photons

Let us consider a single free photon moving along the z direction. Then we can write the four-potential as

$$A^{\mu}(x) = A^{\mu} e^{i\omega(z-t)}, \tag{16}$$

with

$$A^{\mu} = (A_1, A_2, A_3, A_o). \tag{17}$$

The momentum four-vector is clearly

$$P^{\mu} = (0, 0, \omega, \omega) \qquad . \tag{18}$$

Then, the little group applicable to the photon four-potential is generated by

$$J_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 0 & -i & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i \\ 0 & i & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}. \tag{19}$$

These matrices satisfy the commutation relations:

$$[J_3, N_1] = iN_2, \qquad [J_3, N_2] = -iN_1, \qquad [N_1, N_2] = 0, \tag{20}$$

which are identical to those for $E(2)$. From these generators, we can construct the transformation matrix:

$$D(u, v, \alpha) = D(0, 0, \alpha) D(u, v, 0), \tag{21}$$

where

$$D(u, v, 0) = \exp\left(-i(uN_1 + vN_2)\right), \quad D(0, 0, \alpha) = R(a) = \exp\left(-i\alpha J_3\right). \tag{22}$$

We can now expand the above formulas in power series, and the results are

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{23}$$

and

$$D(u, v, 0) = \begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & -v & v \\ u & v & 1 - (u^2 + v^2)/2 & (u^2 + v^2)/2 \\ u & v & -(u^2 + v^2)/2 & 1 + (u^2 + v^2)/2 \end{pmatrix}. \tag{24}$$

When applied to the four-potential, the above D matrix performs a gauge transformation,4 while $R(\alpha)$ is the rotation matrix around the momentum.

The D matrices of Eq.(21) have the same algebraic property as that for the E matrices discussed in Sec. 2. Why, then, do they look so different? In the case of the $O(3)$-like little group, the four-by-four matrices of the little group can be reduced to a block diagonal form consisting of the three-by-three rotation matrix and one-by-one unit matrix [2]. Is it then possible to reduce the D matrices to the form which can be directly compared with the three-by-three E matrices of Sec. 2?

One major problem in bringing the D matrix to the form of the E matrix is that the D matrix is quadratic in the u and v variables. In order to attack this problem, let us impose the Lorentz condition on the four-potential:

$$\frac{\partial}{\partial x^{\mu}} A^{\mu}(x) = P^{\mu} A_{\mu}(x), \tag{25}$$

resulting in $A_3 = A_o$. Since the third and fourth components are identical, the N_1 and N_2 matrices of Eq.(19) can be replaced respectively by

$$N_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}. \tag{26}$$

At the same time, the $D(u, v, 0)$ of Eq.(24) becomes

$$D(u, v, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ u & v & 1 & 0 \\ u & v & 0 & 1 \end{pmatrix}. \tag{27}$$

This matrix has some resemblance to the representation of the cylindrical group given in Eq.(11) [5].

In order to make the above form identical to Eq.(11), we use Dirac's light cone coordinate system in which the combinations $x, y, (z+t)$, and $(z-t)$ are used as the coordinate variables [6]. In this system the four-potential of Eq.(16) is written as

$$A^{\mu} = \left(A_1, A_2, (A_3 + A_o), (A_3 - A_0)\right). \tag{28}$$

The linear transformation from the four-vector of Eq.(16) to the above expression is straightforward. According to the Lorentz condition, the fourth component of the above expression vanishes. We are thus left with the first three components.

During the transformation into the light-cone coordinate system, J_3 remains the same. If we take into account the fact that the fourth component of A now vanishes, N_1 and N_2 become

$$N_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}. \tag{29}$$

As a consequence, $D(u, v)$ takes the form:

$$D(u, v, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ u & v & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{30}$$

and $R(\alpha)$ remains the same as before. It is now clear that the four-by-four representation of the little group is reduced to one three-by-three matrix and one trivial one-by-one matrix. If we use $\hat{J}_3, \hat{N}_1$ and $\hat{N}_2$ for the three-by-three portion of the four-by four J_3, N_1 and N_2 matrices respectively, then

$$\hat{J}_3 = L_3, \quad \hat{N}_1 = Q_1, \quad \hat{N}_2 = Q_2. \tag{31}$$

Now the identification of $E(2)$-like little group with the cylindrical group is complete.

4 The Cylindrical Group as a Contraction of O(3)

The contraction of $O(3)$ to $E(2)$ is well known and discussed widely in the literature.1 The easiest way to understand this procedure is to consider a sphere with large radius, and a small area around the north pole. This area would appear like a flat surface. We can then make Euclidean transformations on this surface, consisting of translations along the x and y directions and rotations around any point within this area. Strictly speaking, however, these Euclidean transformations are $O(3)$ rotations around the x axis, y axis, and around the axis which makes a very small angle with the z axis.

Let us start with the generators of $O(3)$, which satisfy the commutation relations:

$$[L_i, L_j] = i\epsilon_{ijk} L_k \tag{32}$$

generates rotations around the north pole, and its matrix form is given in Eq.(4). L_1 and L_2 take the form

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad L_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}. \tag{33}$$

For the present purpose, we can restrict ourselves to a small region near the north pole, where z is large and is equal to the radius of the sphere R, and x and y are much smaller than the radius. We can then write

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/R \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{34}$$

The column vectors on the left- and right-hand sides are respectively the coordinate vectors on which the $E(2)$ and $O(3)$ transformations are applicable. We shall use the notation A for the three-by-three matrix on the right hand side. In the limit of large R,

$$P_1 = \frac{1}{R} A L_2 A^{-1}, \qquad P_2 = -\frac{1}{R} A L_1 A^{-1}. \tag{35}$$

This procedure leaves L_3 invariant. However, L_1 and L_2 become the P_1 and P_2 matrices discussed in Sec. II. Furthermore, in terms of $P1, P2$ and $L3$, the commutation relations for $O(3)$ given in Eq.(32) become

$$[L_3, P_1] = iP_2, \quad [L_3, P_2] = -iP_1, \quad [P_1, P_2] = -i \left(\frac{1}{R}\right)^2 L_3. \tag{36}$$

In the large$-R$ limit, the commutator $[P_1, P_2]$ vanishes, and the above set of commutators becomes the Lie algebra for $E(2)$.

We have so far considered the area near the north pole where z is much larger than $\sqrt{x^2 + y^2}$. Let us next consider the opposite case, in which $\sqrt{x^2 + y^2}$ is much larger than z. This is the equatorial belt of the sphere. Around this belt, x and y can be written as

$$x = R \cos\phi, \qquad y = R \sin\phi. \tag{37}$$

We can now write

$$\begin{pmatrix} \cos\phi \\ \sin\phi \\ z \end{pmatrix} = \begin{pmatrix} 1/R & 0 & 0 \\ 0 & 1/R & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{38}$$

to obtain the vector space for the cylindrical group dicussed in Sec. 2. The three-by-three matrix on the right-hand side of the above expression is proportional to the inverse of the matrix A given in Eq.(34). Thus in the limit of large R,

$$L_1 = A^{-1} L_1 A, \qquad P_1 = \left(\tfrac{1}{R}\right) A^{-1} L_2 A, P_2 = -\left(\tfrac{1}{R}\right) A^{-1} L_1 A. \tag{39}$$

In terms of L_3, Q_1 and Q_2, the commutation relations for $O(3)$ given in Eq.(32) become

$$[L_3, Q_1] = iQ_2, \qquad [L_3, Q_2] = -iQ_1, \qquad [Q_1, Q_2] = -i \left(\frac{1}{R}\right)^2 L_3. \tag{40}$$

which become the Lie algebra for $E(2)$. The contraction of $O(3)$ to $E(2)$ and to the cylindrical group is illustrated in Fig. 1.

5 E(2)-like Little Group as an Infinite-momentum/zero-mass Limit of the $O(3)$-like Little Group for Massive Particles

If a massive particle is at rest, the symmetry group is generated by the angular momentum operators J_1, J_2 and J_3. If this particle moves along the z direction, J_3 remains invariant, and its eigenvalue is the helicity. However, what happens to J_1 and J_2, particularly in the infinite-momentum limit?

In order to tackle this problem, let us summarize the results of the preceding sections. The generators of the $E(2)$-like little group can be reduced to those of the cylindrical group. The cylindrical group can be obtained from the three-dimensional rotation group through a large-radius approximation. Therefore, if the boost matrix takes a diagonal form as in the case of Eq.(34) or (38), we should be able to obtain N_1 and N_2 by boosting J_2 and J_1 respectively along the z direction [7].

Indeed, in the light-cone coordinate system, the boost matrix takes the form

$$R(P) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & 1/R \end{pmatrix}, \tag{41}$$

with

$$R = \sqrt{\frac{1+\beta}{1-\beta}}, \tag{42}$$

where β is the velocity parameter of the particle. Under this boost, J_3 remains invariant:

$$J_3' = BJ_3B^{-1} = J_3. \tag{43}$$

J_1 and J_2 in the light-cone coordinate system take the form

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i \\ 0 & i & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i \\ 0 & i & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}. \tag{44}$$

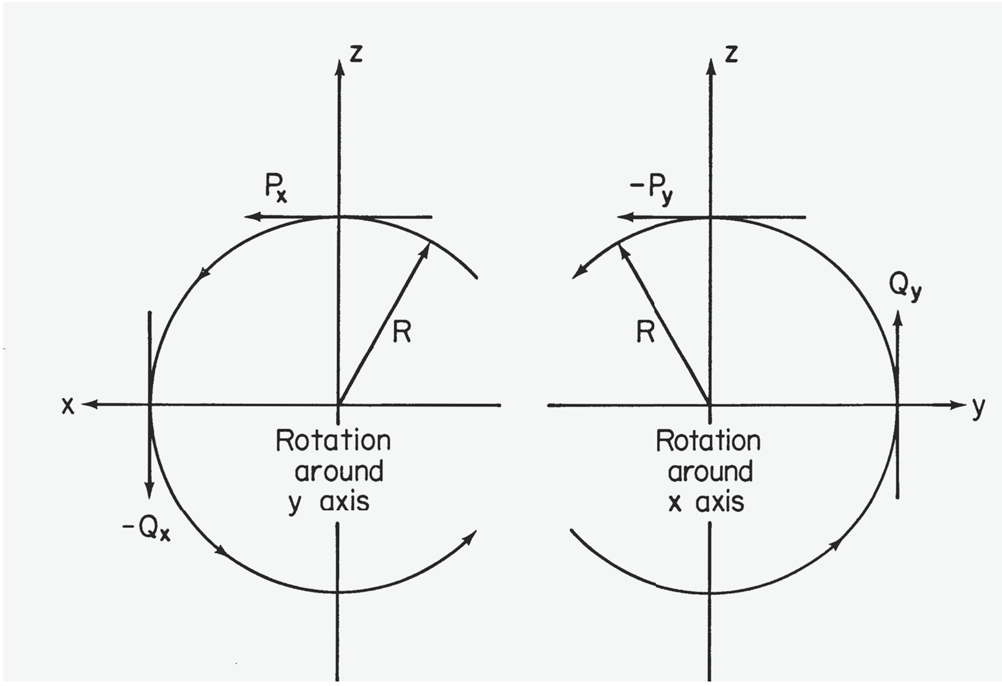


Figure 1: Contraction of the three-dimensional rotation group to the two-dimensional Euclidean group and to the cylindrical group. The rotation around the z axis remains unchanged as the radius becomes large. In the case of $E(2)$, rotations around the y and x axes become translations in the x and $-y$ directions respectively within a flat area near the north pole. In the case of the cylindricalgroup, the rotations around the y and x axes result in translations in the negative and positive z directions respectively within a cylindrical belt around the equator.

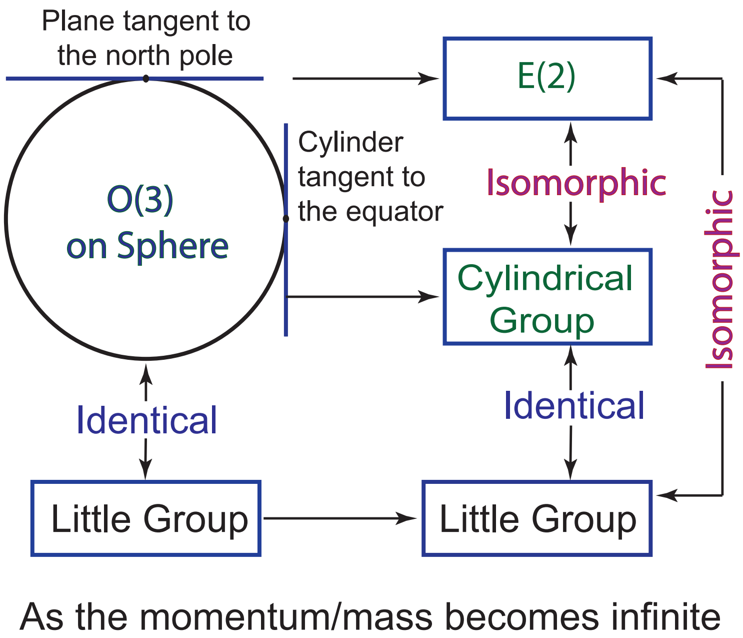


Figure 2: $E(2)$, the $E(2)$-like little group for massless particles, and the cylindrical group. The correspondence between $E(2)$ and the $E(2)$-like little group is isomorphic but not identical. The cylindrical group is identical to the $E(2)$-like little group. Both $E(2)$ and the cylindrical group can be regarded as contractions of $O(3)$ in the large-radius limit. The Lorentz boost of the $O(3)$-like little group for a massive particle at rest to the $E(2)$-like little group for a massless particle is exactly the same as the contraction of $O(3)$ to the cylindrical group. The radius of the sphere in this case can be identified as $\sqrt{(1+\beta)/(1-\beta)}$.

If we boost this massive particle along the z direction, the boosted J_1 and J_2 become

$$J_1' = BJ_1B^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i/R & i/R \\ 0 & iR & 0 & 0 \\ 0 & -iR & 0 & 0 \end{pmatrix},$$

$$J_2' = BJ_2B^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -i/R & i/R \\ 0 & 0 & 0 & 0 \\ iR & 0 & 0 & 0 \\ -iR & 0 & 0 & 0 \end{pmatrix}. \tag{45}$$

Because of the Lorentz condition, the iR terms in the fourth column of the above matrices can be dropped. Therefore, in the large-R limit which is the limit of large momentum,

$$N_1 = -\frac{1}{R}J_2', \qquad N_2 = \frac{1}{R}J_1'. \tag{46}$$

where N_1 and N_2 are given in Eq.(30). This completes the proof that the gauge degrees of freedom in the $E(2)$-like little group for photons are Lorentz-boosted rotational degrees of freedom. The limiting process is the same as the contraction of the three-dimensional rotation group to the cylindrical group.

Concluding Remarks

The isomorphism between the two-dimensional Euclidean group and the little group for massless particles is well known and well understood. However, the isomorphism in this case does not mean that they are identical. We have shown in this paper that the $E(2)$-like little group can be reduced to the identity group and the cylindrical group which is isomorphic to $E(2)$. As in the case of $E(2)$, we can obtain the cylindrical group by contracting the three-dimensional rotation group. This contraction procedure is identical to the Lorentz boost of the $O(3)$-like little group for a massive particle at rest to the $E(2)$-like little group for a massless particle. The result of the present paper is summarized in Fig. 2.

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