Wigner's little group for massless particles

If we use the four-vector convention $x^{\mu} = (x, y, z, t)$, the generators of rotations around and boosts along the z axis take the form

$$J_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \tag{1}$$

respectively. We can also write the four-by-four matrices for J_1 and J_2 for the rotations around the x and y directions, as well as K_1 and K_2 for Lorentz boosts along the x and y directions respectively. These six generators satisfy the following set of commutation relations.

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \qquad [J_i, K_j] = i\epsilon_{ijk} K_k, \qquad [K_i, K_j] = -i\epsilon_{ijk} J_k. \tag{2}$$

This closed set of commutation relations is called the Lie algebra of the Lorentz group. The three J_i operators constitute a closed subset of this Lie algebra. Thus, the rotation group is a subgroup of the Lorentz group.

In addition, Wigner in 1939 [1] considered a group generated by

$$J_3, \qquad N_1 = K_1 - J_2, \qquad N_2 = K_2 + J_1. \tag{3}$$

These generators satisfy the closed set of commutation relations

$$[N_1, N_2] = 0, \qquad [J_3, N_1] = iN_2, \qquad [J_3, N_2] = -iN_1. \tag{4}$$

As Wigner observed in 1939 [1], this set of commutation relations is just like that for the generators of the two-dimensional Euclidean group with one rotation and two translation generators, as illustrated in Fig. 1. However, the question is what aspect of the massless particle can be explained in terms of this two-dimensional geometry.

$$L_3 = \begin{pmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5}$$

applicable to the three-dimensional (x, y, z) space.

The L_3 matrix generate rotations around the z axis. As for the P_1 and P_2 matrices, they commute with each other, and

$$\exp\{-(uP_1 + vP_2)\} = \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

If this matrix is applied to the (x, y, z) space,

$$\begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = . \begin{pmatrix} x + uz \\ y + yz \\ z \end{pmatrix}. \tag{7}$$

Thus P_1 and P_2 generate translations along the x and y directions respectively.

Thus, the three matrices L_3, P_1, and P_3 are the generators of the two-dimensional Euclidean group. Furthermore, they satisfy the following closest set of commutation relations.

$$[P_1, P_2] = 0, \quad [L_3, P_1] = iP_2, \quad [L_3, P_2] = -iP_1. \tag{8}$$

Thus, the three matrices L_3, P_1, and P_3 are the generators of the two-dimensional Euclidean group, or the E(2) group. Let us replace these generators by J_3, N_1, and N_2 respectively. This set of commentators become that for the little group of the massless particle given in Eq.(8).

Wigner observed this in his 1939 paper. Thus the little group for massless particles is locally isomorphic to E(2). However, what physical transformations do these translation-like N_1 and N_2 perform? Wigner did not provide the answer to this question.

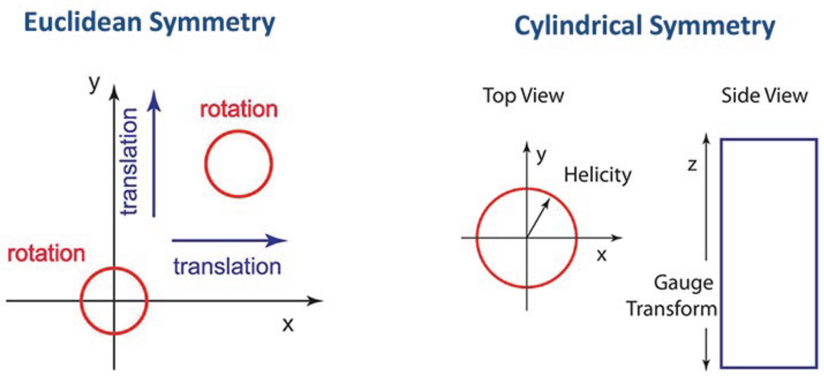

Figure 1: Transformations of the $E(2)$ group and the cylindrical group. They share the same Lie algebra, but only the cylindrical group leads to a geometrical interpretation of the gauge transformation.

Without realizing this as a Wigner problem, Kuperzstych in 1976 noted that they generate gauge transformations [2]. Then what happens to the E(2) group with two-independent translational degrees of freedom? In order to answer this question, we note the commutation relations for the E(2) remains invariant under the following replacements.

$$P_1 \rightarrow Q_1 = -P_1^{\dagger}, \qquad P_2 \rightarrow Q_2 = -P_2^{\dagger}, \tag{9}$$

with

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}. \tag{10}$$

These generators lead to the

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & v & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = . \begin{pmatrix} x \\ y \\ z + ux + vy \end{pmatrix}. \tag{11}$$

Indeed this transformation leave the x and y variable invariant, but changes the z component.

Indeed, this question has a stormy history, and was not answered until 1987. In their paper of 1987 [3], Kim and Wigner considered the surface of a circular cylinder as shown in Fig. 1.

For this cylinder, rotations are possible around the z axis. It is also possible to make translations along the z axis as shown in Fig. 1. We can write these generators as

$$L_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \tag{12}$$

applicable to the three-dimensional space of (x, y, z). They then satisfy the closed set of commutation relations

$$[Q_1, Q_2] = 0, \qquad [L_3, Q_1] = iQ_2, \qquad [L_3, Q_2] = -iQ_1. \tag{13}$$

which becomes that of Eq.(8) when Q_1, Q_2, and L_3 are replaced by N_1, N_2, and J_3 of Eq.(3) respectively. Indeed, this cylindrical group is locally isomorphic to Wigner's little group for massless particles.

Let us go back to the generators of Eq.(3). The role of J_3 is well known. It is generates rotations around the momentum and corresponds to the helicity of the massless particle. The N_1 and N_2 matrices take the form

$$N_1 = \begin{pmatrix} 0 & 0 & -i & i \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & i \\ 0 & i & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}. \tag{14}$$

The transformation matrix is

$$\begin{aligned} D(u, v) &= \exp\{-i\,(uN_1 + vN_2)\} \\ &= \begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & -v & v \\ u & v & 1 - (u^2 + v^2)/2 & (u^2 + v^2)/2 \\ u & v & -(u^2 + v^2)/2 & 1 + (u^2 + v^2)/2 \end{pmatrix}. \end{aligned} \tag{15}$$

If this matrix is applied to the electromagnetic wave propagating along the z direction,

$$A^{\mu}(z, t) = (A_1, A_2, A_3, A_0)e^{i\omega(z-t)}, \tag{16}$$

which satisfies the Lorentz condition $A_3 = A_0$, the $D(u,v)$ matrix can be reduced to

$$D(u,v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ u & v & 1 & 0 \\ u & v & 0 & 1 \end{pmatrix}. \tag{17}$$

If $A_3 = A_0$, the four-vector (A_1, A_2, A_3, A_3) can be written as

$$(A_1, A_2, A_3, A_0) = (A_1, A_2, 0, 0) + \lambda(0, 0, \omega, \omega), \tag{18}$$

with $A_3 = \lambda\omega$. The four-vector $(0, 0, \omega, \omega)$ represents the four-momentum. If the D matrix of Eq.(17) is applied to the above four vector, the result is

$$(A_1, A_2, A_3, A_0) = (A_1, A_2, 0, 0) + \lambda'(0, 0, \omega, \omega), \tag{19}$$

with $\lambda' = \lambda + (1/\omega)(uA_1 + vA_3)$. Thus the D matrix performs a gauge transformation when applied to the electromagnetic wave propagating along the z direction.

References

[1] **E. Wigner, *On unitary Representations of the Inhomogeneous Lorentz Group*, Ann. Math. 40, 149 - 204 (1939).**

[2] **J. Kuperzstych, *Is there a Link between Gauge Invariance, Relativistic Invariance and Electron Spin?* Nuovo Cimento 31B, 1 - 11 (1976).**

[3] **Y. S. Kim and E. P. Wigner, *Cylindrical group and massless particles*, J. Math. Phys. 28, 1175 - 1179 (1987).**