A Historical Note on Group Contractions

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I wish to welcome you to this meeting and hope that you will find it interesting and rewarding. It was organised, essentially, following the suggestions of Drs. Kim, Pogosyan, Gromov and Duru, that we should once more bring together people interested in various aspects of deformations of Lie groups, including quantum groups and the simplest case of contractions. I am happy to see that quite a few researchers have found time to come and join us here. The organising committee has asked me to give an introductory talk, as one of the two co-authors of one of the first papers on group contractions. Actually, since almost half a century passed since this paper was published in June 1953, 44 years ago to be exact, it can be considered as part of science history. Furthermore, as I am nowadays more interested in writing memoirs, rather than following current research developments, I shall tell you the story of this article. It has some interesting and instructive aspects from both scientific and human points of view and it will be a tribute to the memory of Prof. Wigner. In the beginning I must tell you that I completed all the requirements for a Ph.D. degree at CalTech in October of 1951, and then came to Princeton University as a visiting research fellow, to spend something like six months doing research, before going back to my native country, Turkey. My thesis was a phenomenological piece of work, applying the shower theory of Oppenheimer and Snyder to cosmic ray bursts observed by the team of Neher from CalTech. It had nothing to do with group theory. But my reason for coming to Princeton was to do something in group theory, which fascinated me, although at that time I knew very little about it. I had learned from my elders at CalTech that E.P.Wigner was a world famous authority in group theory (as in many other areas). During the previous summer I had visited him at the University of Wisconsin where he was working and told him of my wish to come to Princeton following the completion of my Ph.D. work at CalTech. After receiving a favourable letter from Prof. Christy, my thesis advisor, he invited me to spend the new academic year at Princeton. In our first meeting in his office at Palmer Laboratory in Princeton, I gave him the details of my education at CalTech and expressed my desire to work with him on a problem in group theory. He asked me a few questions in group theory and found out that my knowledge was very meagre in this field. He did not say anything about whether or not he was going to give me any problems , but at the end of our conservation , he very kindly invited me to have lunch at the cafeteria.

While we were waiting in line for our turn to take our meals, another member of the physics department came along and shook hands with Wigner. He introduced me to the newcomer who asked me what I did at CalTech. I replied by saying, "Prof. Christy was kind enough to propose a problem involving cosmic ray showers which provided indirect evidence for the existence of neutral pi mesons." I was giving this information automatically, without any afterthought, but at the same time I noticed that Wigner was touched as if I were implying that so far Princeton had not been as kind to me as CalTech.

Whatever may be the reason, after lunch Wigner took me back to his office and proposed to me to work on determining the unitary irreducible representations of the inhomogeneous Galilee group. As I looked somewhat perplexed, he explained that this would be a way to find in quantum mechanics, the most general equations of motion for a non-relativistic particle and that I could follow the same methods that he had developed previously for the Lorentz group.

Prof. Wigner's classical paper on the unitary irreducible representations of the inhomogeneous Lorentz group or the Poincaré group to be more precise, had appeared in 1939 [1] but its importance was being realised only in the fifties. He gave me a reprint and then together we went to the library, where for my reading we collected a few books on group theory. His own book was available only in German. He asked me whether I could read German and when I replied saying that I could do it with the help of a dictionary, he said, "Well, it will be hard going, but you can do it". He then added a very interesting comment: "Of course, what is really important is imagination!" More than a month went by while I studied the essentials of group theory and read several times his paper on the Lorentz group. Then I started to work on the Galilei representations, benefiting from his advice all the time. I suspect that he already knew how these representations will turn out to be, but he did not say it. He only guided me in determining them.

There happens to be four classes of unitary irreducible representations. The most general one may be written in the following form which is diagonal in space translations and boosts (special Galilei transformations) and where the wave functions depend on two vectors $\vec{p}$, $\vec{q}$ [3]:

$$\begin{aligned} T(\vec{a})\psi(\vec{p},\vec{q}\,) &= e^{i\vec{p}\cdot\vec{a}}\;\psi(\vec{p},\vec{q}\,) \\ G(\vec{v})\psi(\vec{p},\vec{q}\,) &= e^{i\vec{q}\cdot\vec{v}}\;\psi(\vec{p},\vec{q}\,) \\ \Theta(b)\psi(\vec{p},\vec{q}\,) &= \psi(\vec{p},\vec{q}-b\vec{p}\,) \\ O(R)\psi(\vec{p},\vec{q}\,) &= \psi(R^{-1}\vec{p},R^{-1}\vec{q}\,) \end{aligned} \tag{1}$$

In this most general case, the wave functions are characterised by the positive numbers P and S where:

$$P^2 = \vec{p}\cdot\vec{p} \quad , \quad S = \;|\vec{p}\times\vec{q}\,|\;. \tag{2}$$

$T(\vec{a})$ is the operator for a space translation by the vector $\vec{a}$, $G(\vec{v})$ is the operator for a boost, $\vec{x}' = \vec{x} - \vec{v}t$, where $\vec{v}$ is the constant velocity vector, $\Theta(b)$ is the operator for a translation in time by the amount b and $O(R)$ is the operator for a rotation denoted by the matrix R.

We could define a scalar product in the Hilbert space of the functions $\psi(\vec{p},\vec{q})$ by the expression:

$$(\psi,\phi) = \int \psi(\vec{p},\vec{q})\ \phi(\vec{p},\vec{q})\ \delta(\vec{p}^2 - P^2)\ \delta(\ |\ \vec{p}\times\vec{q}\ \ | - S\ )\ d\vec{p}\, d\vec{q} \quad . \tag{3}$$

The next step in Wigner's programme was to find a physical meaning for these representations. Following the example of the Lorentz case, we assumed that they would represent a single particle. But here we found an unexpected result. If a representation ψ is to describe a particle localised at $x = y = z = 0$ at time $t = 0$, it will be orthogonal to the function $T(\vec{a})\psi$, for $\vec{a} \neq 0$ [2]. In other words, we must have

$$(T(\vec{a})\psi\ ,\ \psi) = 0 \tag{4}$$

or,

$$\int e^{i\vec{p}\cdot\vec{a}}\, |\psi(\vec{p},\vec{q})|^2\ \ \delta(\vec{p}^2 - P^2)\ \delta(\ |\ \vec{p}\times\vec{q}\ \ | - S\ )\ d\vec{p}\, d\vec{q} = 0 \tag{5}$$

But this can be reduced to

$$\frac{\sin Pa}{Pa} = 0 \tag{6}$$

which is not possible. Thus, we concluded that these representations could not describe localised particles.

We have also looked for states with definite velocity, but could not find them either. So the unescapable conclusion was that the irreducible unitary representations of the inhomogeneous Galilei group could not be interpreted as describing non-relativistic particles. On the other hand, it was known that the free particle solutions of the non-relativistic Schrödinger equation corresponded to the up-to-a factor representations of the inhomogeneous Galilei group.

The classical argument of Wigner went as follows :

In quantum mechanics, the transition probability between two states ψ , ϕ defined as $|(\psi,\phi)|^2$ must be invariant with respect to the change of reference frames :

$$|(\phi_g,\psi_g)|^2 = |(\phi_{g'},\psi_{g'})|^2 \tag{7}$$

where g, g' denote two different reference frames.

We can define a linear, unitary operator $D(N)$ such that, $\phi_{g'} = D(N)\phi_g$ where N is the transformation which carries g into g':

$$g' = Ng \tag{8}$$

The operator $D(N)$ is determined by the physical theory only up to a constant of modulus unity which can depend on g and g'. So, the $D(N)$ form a representation up to a factor of the invariance group :

$$D(N_1)D(N_2) = \omega(N_1,N_2)D(N_1N_2) \tag{9}$$

where ω is a complex number whose phase may depend on N_1, N_2 but its modulus is unity, i.e.:

$$\omega = e^{iu(N_1,N_2)} \tag{10}$$

In the case of the Poincaré group, Wigner was able to assign a definite phase to each operator $D(N)$, leaving only the sign undetermined. He thus obtained, for the normalised operators $U(P)$,

$$U(P_1)U(P_2) = \mp U(P_1 P_2) \tag{11}$$

Applying similar arguments to the Galilei group, I obtained the following relation:

$$U(G_1)U(G_2) = \mp \exp\left\{-2\pi i A(\vec{a_2}\cdot\vec{v_1} + \frac{1}{2}\, b_2\, v_1^2)\right\} U(G_1 G_2) \tag{12}$$

where A is an arbitrary constant. This is essentially the representation formed by the plane wave solutions of the non-relativistic Schrödinger equation for a particle with mass m, spin zero and momentum $\vec{p}$:

$$\psi(\vec{p}) = \exp\left\{\frac{2\pi i}{h}(\vec{p}\cdot\vec{x} - \frac{p^2}{2m}\, t)\right\} \tag{13}$$

which yields, taking the positive sign for spin zero,

$$U(G_1)U(G_2)\psi(\vec{p}) = +\exp\left\{-2\pi i\frac{m}{h}(\vec{a_2}\cdot\vec{v_1} + \frac{1}{2}\, b_2\, v_1^2)\right\} \tag{14}$$

so that we have here $A = -\frac{m}{h}$ [4].

When I reached this point, the original programme proposed to me by Wigner was completed and I started to write the paper on the Galilei representations. But a question remained: How is it that, the true representations of the Poincare group have a physical meaning while those of the Galilei group do not? Or, in other words, how does the physical meaning disappear when one goes over from the Poincaré group to the Galilei group? We thought that at least a partial answer could be obtained by looking at the limits for infinite light velocity of the specific unitary representations of the Poincaré group obtained by Wigner. The idea was to add an appendix to our Galilei paper, giving the results of this limiting process.

However, when I tried to take the limits of the unitary representations of the Poincaré group, the outcome became incomprehensible. The limiting process gave a finite answer in some cases, but vanished altogether in other cases. After we struggled for a couple of weeks without obtaining consistent results, Wigner had the bright idea of separating the problem into its essential components. He said : "Let us first look at the limit of the group, understand what happens there, and then consider the limits of the representations." This approach gave the clue for solving our difficulties. When we considered a singular transformation on the infinitesimal generators I_λ of the original n-parameter Lie group, in the following form given by Wigner,

$$\begin{array}{ccc} J_{1\nu} = I_{1\nu} & \text{where} & \nu = 1, 2, \ldots, r \\ J_{2\mu} = \epsilon I_{2\mu} & & \mu = 1, 2, \ldots, n-r \end{array} \tag{15}$$

and took the limit for $\epsilon \to 0$, everything became clear and simple. Let the Lie algebra of the original group be given by:

$$[I_\kappa\ ,\ I_\lambda] = \sum_{\tau=1}^{n} c_{\kappa\lambda}^{\tau}\, I_\tau \tag{16}$$

Applying the above-mentioned transformation, we obtain the new algebra as:

$$\begin{aligned} [J_{1\nu}\ ,\ J_{1\mu}] = [I_{1\nu}\ ,\ I_{1\mu}] &= \sum_{\kappa=1}^{r} c_{1\nu,1\mu}^{1\kappa}\, I_{1\kappa} + \sum_{\lambda=1}^{n-r} c_{1\nu,1\mu}^{2\lambda}\, I_{2\lambda} \\ &= \sum_{\kappa=1}^{r} c_{1\nu,1\mu}^{1\kappa}\, J_{1\kappa} + \frac{1}{\epsilon}\sum_{\lambda=1}^{n-r} c_{1\nu,1\mu}^{2\lambda}\, J_{2\lambda}, \\ [J_{1\nu}\ ,\ J_{2\mu}] = \epsilon\,[I_{1\nu}\ ,\ I_{2\mu}] &= \epsilon \sum_{\kappa=1}^{r} c_{1\nu,2\mu}^{1\kappa}\, J_{1\kappa} + \sum_{\lambda=1}^{n-r} c_{1\nu,2\mu}^{2\lambda}\, J_{2\lambda}, \\ [J_{2\nu}\ ,\ J_{2\mu}] = \epsilon^2\,[I_{2\nu}\ ,\ I_{2\mu}] &= \epsilon^2 \sum_{\kappa=1}^{r} c_{2\nu,2\mu}^{1\kappa}\, J_{1\kappa} + \epsilon \sum_{\lambda=1}^{n-r} c_{2\nu,2\mu}^{2\lambda}\, J_{2\lambda}. \end{aligned} \tag{17}$$

Now, taking the limit for $\epsilon \to 0$, we see that, the new algebra will give finite results only if $c_{1\nu,1\mu}^{2\lambda} = 0$, in other words, only if the infinitesimal generators $I_{1\nu}$ span a subalgebra. In this case, the new Lie algebra will have the form:

$$\begin{aligned} [J_{1\nu}\ ,\ J_{1\mu}] &= \sum_{\kappa=1}^{r} c_{1\nu,1\mu}^{1\kappa}\, J_{1\kappa}, \\ [J_{1\nu}\ ,\ J_{2\mu}] &= \sum_{\kappa=1}^{n-r} c_{1\nu,2\mu}^{2\kappa}\, J_{2\kappa}, \\ [J_{2\nu}\ ,\ J_{2\mu}] &= 0. \end{aligned} \tag{18}$$

The conclusion is that if the operators $I_{1\mu}$, formed a subgroup S originally, we would always obtain a new group by taking the limit for $\epsilon \to 0$. The operators $I_{2\mu}$ lead to an invariant abelian subgroup. The subgroup S with respect to which the limiting operation is carried out, is isomorphic with the factor group of this invariant subgroup. In a sense, we are contracting the original group over the subgroup S which remains invariant, and this is why we named this process a contraction. Once we saw this picture, it was clear that the process could be applied to many groups, one of the simplest examples being the three dimensional rotation group. Contracting $O(3)$, we obtain $E(2)$, the two dimensional Euclidean group, rotations and displacements in the plane. On the other hand, what we considered was such a simple process that for a while, we did not believe that it was something unknown until that time. Wigner asked Bargmann about it. Bargmann, who knew the literature extensively was surprised, but admitted that he had not seen this process defined previously. I looked at the collected works of Sophus Lie, but did not find it mentioned, so we decided that it was something original. Wigner, then told me that it would not be correct to put it as an appendix to the paper on the Galilei representations, since it seemed to have a more general area of application. Bargmann noticed later that, Segal, in a paper published in 1951 [5], had considered a sequence of Lie groups which converges to another Lie group. When we looked at the representations of the contracted group, the situation which had puzzled us before, became resolved. If we apply the contraction directly to the infinitesimal operators of the representation, we do obtain a representation of the contracted group; but in general it will not be faithful. It would be a representation isomorphic to the subgroup with respect to which contraction is taken or, in other words, it will be a representation of the factor group of the invariant subgroup. In order to obtain a faithful representation of the contracted group, one can either carry out first an ϵ-dependent transformation on the $I_{2\nu}$ or take the operators $J_{2\nu}$ which correspond to different representations, i.e., go to higher and higher dimensional representations. Using these methods, I was able to contract the representations of the Poincaré group and show in fact that, the true representations of the Galilei group for which we did not find any physical meaning, are the limits of the unphysical representations of the Poincaré group corresponding to space-like or null momenta. On the other hand, those representations with time-like momenta can only be contracted to up-to-a factor representations of the Galilei group, corresponding to the solutions of the Schrödinger equation. After we reached that stage, I returned to Turkey and the paper on contractions was completed through correspondence [6]. To complete the human side let me make two final observations. The story shows, at first, how we came to the simple idea of group contraction by a very roundabout way and secondly, how I was lucky to have asked Wigner to give me a problem at the right time.

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