Illustrative Example of Feynman's Rest of the Universe*

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Abstract

Coupled harmonic oscillators occupy an important place in physics teaching. It is shown that they can be used for illustrating an increase in entropy caused by limitations in measurement. In the system of coupled oscillators, it is possible to make the measurement on one oscillator while averaging over the degrees of freedom of the other oscillator without measuring them. It is shown that such a calculation would yield an increased entropy in the observable oscillator. This example provides a clarification of Feynman's rest of the universe.

I. INTRODUCTION

Because of its mathematical simplicity, the harmonic oscillator provides soluble models in many branches of physics. It often gives a clear illustration of abstract ideas. In his book on statistical mechanics [1], Feynman makes the following statement about the density matrix. *When we solve a quantum-mechanical problem, what we really do is divide the universe into two parts - the system in which we are interested and the rest of the universe. We then*

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usually act as if the system in which we are interested comprised the entire universe. To motivate the use of density matrices, let us see what happens when we include the part of the universe outside the system. The purpose of this paper is to study Feynman's rest of the universe and related problems of current interest using a pair of coupled oscillators.

Let us start with a single oscillator in its ground state. Quantum mechanically, there are many kinds of excitations of the oscillator, and three of them are familiar to us. First, it can be excited to a state with a definite energy eigenvalue. We obtain the excited-state wave functions by solving the eigenvalue problem for the Schrödinger equation, and this procedure is well known. Second, the oscillator can go through coherent excitations. The ground-state oscillator can be excited to a coherent or squeezed state. During this process, the minimum uncertainty of the ground state is preserved. The coherent or squeezed state is not in an energy eigenstate. This kind of excited state plays a central role in coherent and squeezed states of light which have recently become a standard item in quantum mechanics.

Third, the oscillator can go through thermal excitations. This is not a quantum excitation, but is a statistical ensemble. We cannot express a thermally excited state by making linear combinations of wave functions. We should treat this as a canonical ensemble. In order to deal with this thermal state, we need a density matrix [1–3].

As for the density matrix, this was first introduced in 1932 by von Neumann [4]. Since quantum mechanics is based on the inherent limitation of measurements, the density matrix depends heavily how measurements are taken in laboratories. Sometimes, we measure all the variables allowed by quantum mechanics and sometimes we do not. Given the density matrix of the system, in order to take into account the effect of the unmeasured variables, we sum over them [4–7].

Feynman in his book packs the variables we do not measure into the "rest of the universe" (Although we will not do so repeatedly in this paper, the quotation marks are implicit around this key phrase throughout the paper). He is of course interested in the effect of the rest of the universe on the system in which we are able to make measurements. In this paper, we study a set of two coupled oscillators to illustrate Feynman's rest of the universe. In this system, the first oscillator is the universe in which we are interested, and the second oscillator is the rest of the universe.

There will be no effects on the first oscillator if the system is decoupled. Once coupled, the problem becomes much more complicated than those treated in standard textbooks, and requires a group theoretical treatment [8]. But, in this paper, we shall avoid group theory and concentrate on the physical problems we are set to resolve.

In Sec. II, we reformulate the classical mechanics of two coupled oscillators. The symmetry operations include rotations and squeezes in the two-dimensional coordinate system of two oscillator coordinates. In Sec. III, this symmetry property is extended to the quantum mechanics of the coupled oscillators. In Sec. IV, we use the density matrix to study the entropy of the system when our ignorance of the unobserved variable suggests that we average over it. In Sec. V, we use the Wigner phase-space distribution function to see the effect of the averaging on the uncertainty relation in the observable world. In Sec. VI, it is shown that the system of two coupled oscillators can serve as an analog computer for many of the physical theories and models of current interest.

II. COUPLED OSCILLATORS IN CLASSICAL MECHANICS

Two coupled harmonic oscillators serve many different purposes in physics. It is well known that this oscillator problem can be formulated into a problem of a quadratic equation in two variables. The diagonalization of the quadratic form includes a rotation of the coordinate system. However, the diagonalization process requires additional transformations involving the scales of the coordinate variables [9]. Recently, it was found that the mathematics of this procedure can be as complicated as the group theory of Lorentz transformations in a six dimensional space with three spatial and three time coordinates [8].

In this paper, we avoid group theory and use a set of two-by-two matrices in order to understand what Feynman says in his book. This set, of course, includes rotation matrices. In addition, the set will include scale transformations on the position and momentum variables. Let us consider a system of coupled oscillators. The Hamiltonian for this system is

$$H = \frac{1}{2}\left\{\frac{1}{m_1}p_1^2 + \frac{1}{m_2}p_2^2 + Ax_1^2 + Bx_2^2 + Cx_1x_2\right\}. \tag{2.1}$$

It is possible to diagonalize by a single rotation the quadratic form of x_1 and x_2. However, the momentum variables undergo the same rotation. Therefore, the uncoupling of the potential energy by rotation alone will lead to a coupling of the two kinetic energy terms.

In order to avoid this complication, we have to bring the kinetic energy portion into a rotationally invariant form. For this purpose, we will need the transformation

$$\begin{pmatrix} p_1' \\ p_2' \end{pmatrix} = \begin{pmatrix} (m_2/m_1)^{1/4} & 0 \\ 0 & (m_1/m_2)^{1/4} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \tag{2.2}$$

This transformation will change the kinetic energy portion to

$$\frac{1}{2m}\left\{p_1'^2 + p_2'^2\right\} \tag{2.3}$$

with $m = (m_1m_2)^{1/2}$. This scale transformation does not leave the x_1 and x_2 variables invariant. If we insist on canonical transformations [8], the transformation becomes

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} (m_1/m_2)^{1/4} & 0 \\ 0 & (m_2/m_1)^{1/4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2.4}$$

The scale transformations on the position variables are inversely proportional to those of their conjugate momentum variables. This is based on the Hamiltonian formalism where the position and momentum variables are independent variables.

On the other hand, in the Lagrangian formalism, where the momentum is proportional to the velocity which is the time derivative of the position coordinate, we have to apply the same scale transformation for both momentum and position variables [9]. In this case, the scale transformation takes the form

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} (m_2/m_1)^{1/4} & 0 \\ 0 & (m_1/m_2)^{1/4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2.5}$$

With Eq.(2.2) for the momentum variables, this expression does not constitute a canonical transformation.

The canonical transformation leads to a unitary transformation in quantum mechanics. The issue of non-canonical transformation is not yet completely settled in quantum mechanics and is still an open question [8]. In either case, the Hamiltonian will take the form

$$H = \frac{1}{2m}\left\{p_1^2 + p_2^2\right\} + \frac{1}{2}\left\{Ax_1^2 + Bx_2^2 + Cx_1x_2\right\}, \tag{2.6}$$

Here, we have deleted for simplicity the primes on the x and p variables.

We are now ready to decouple this Hamiltonian by making the coordinate rotation:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2.7}$$

Under this rotation, the kinetic energy portion of the Hamiltonian in Eq.(2.6) remains invariant. Thus we can achieve the decoupling by diagonalizing the potential energy. Indeed, the system becomes diagonal if the angle α becomes

$$\tan 2\alpha = \frac{C}{B - A}. \tag{2.8}$$

This diagonalization procedure is well known. What is new in this note is the introduction of the new parameters K and η defined as

$$\begin{aligned} K &= \sqrt{AB - C^2/4}, \\ \exp(\eta) &= \frac{A + B + \sqrt{(A-B)^2 + C^2}}{\sqrt{4AB - C^2}}. \end{aligned} \tag{2.9}$$

In terms of this new set of variables, the Hamiltonian can be written as

$$H = \frac{1}{2m}\left\{p_1^2 + p_2^2\right\} + \frac{K}{2}\left\{e^{2\eta}y_1^2 + e^{-2\eta}y_2^2\right\}, \tag{2.10}$$

with

$$\begin{aligned} y_1 &= x_1\cos\alpha - x_2\sin\alpha, \\ y_2 &= x_1\sin\alpha + x_2\cos\alpha. \end{aligned} \tag{2.11}$$

This completes the diagonalization process. The normal frequencies are

$$\omega_1 = e^{\eta}\omega, \qquad \omega_2 = e^{-\eta}\omega, \tag{2.12}$$

with

$$\omega = \sqrt{\frac{K}{m}}. \tag{2.13}$$

This relatively new set of parameters serves a useful purpose when we discuss canonical transformations of the two-oscillator system [8]. This set will also serve useful purposes when we discuss Feynman's rest of the universe.

Let us go back to Eq.(2.6) and Eq.(2.8). If $\alpha = 0$, C becomes zero and the oscillators become decoupled. If $\alpha = 45^o$, then $A = B$, which means that the system consists of two identical oscillators coupled together by the C term. In this case,

$$\exp(\eta) = \sqrt{\frac{2A + C}{2A - C}}. \tag{2.14}$$

Thus η measures the strength of the coupling. The mathematics becomes very simple for $\alpha = 45^o$, and this simple case can be applied to many physical problems. We shall discuss this special case in Sec. VI.

III. QUANTUM MECHANICS OF COUPLED OSCILLATORS

If y_1 and y_2 are measured in units of $(mK)^{1/4}$, the ground-state wave function of this oscillator system is

$$\psi_\eta(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{2}(e^\eta y_1^2 + e^{-\eta} y_2^2)\right\}, \tag{3.1}$$

The wave function is separable in the y_1 and y_2 variables. However, for the variables x_1 and x_2, the story is quite different. For simplicity, we are using here the unit system where $\hbar = 1$.

The key question is how quantum mechanical calculations in the world of the observed variable are affected when we average over the other, unobserved variable. This effect is not trivial. Indeed, the x_2 space in this case corresponds to Feynman's rest of the universe, if we only consider quantum mechanics in the x_1 space. We shall discuss in this paper how we can carry out a quantitative analysis of Feynman's rest of the universe.

Let us write the wave function of Eq.(3.1) in terms of x_1 and x_2, then

$$\psi_\eta(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{2}\left[e^\eta (x_1 \cos\alpha - x_2 \sin\alpha)^2 + e^{-\eta}(x_1 \sin\alpha + x_2 \cos\alpha)^2\right]\right\}. \tag{3.2}$$

If $\eta = 0$, this wave function becomes

$$\psi_0(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{2}(x_1^2 + x_2^2)\right\}. \tag{3.3}$$

It is possible to obtain Eq.(3.2) from Eq.(3.3) by a unitary transformation [8]. The content of the unitary transformation is nothing more than an expansion in complete orthonormal states. The expression given in Eq.(3.3) is a product of two ground-state wave functions. For each of x_1 and x_2, there is a complete set of orthonormal wave functions. For those wave functions, let us use $\phi_k(x_1)$ and $\phi_n(x_2)$. Then we should be able to write Eq.(3.2) as

$$\sum_{m_1 m_2} A_{m_1 m_2}(\alpha, \eta) \phi_{m_1}(x_1) \phi_{m_2}(x_2), \tag{3.4}$$

where $\phi_m(x)$ is the mth excited-state oscillator wave function. The coefficients $A_{m_1 m_2}(\alpha, \eta)$ satisfy the unitarity condition

$$\sum_{m_1 m_2} |A_{m_1 m_2}(\alpha, \eta)|^2 = 1. \tag{3.5}$$

It is possible to carry out a similar expansion in the case of excited states.

The question then is what lessons we can learn from the situation in which we average over the x_2 variable. In order to study this problem, we use the density matrix and the Wigner phase-space distribution function.

IV. DENSITY MATRIX AND ENTROPY

Density matrices play very important roles in quantum mechanics and statistical mechanics, especially when not all measurable variables are measured in laboratories. But they are not fully discussed in the existing textbooks. On the other hand, since there are books and review articles on this subject [6,3,7], it is not necessary here to give a full-fledged introduction to the theory of density matrices.

In this paper, we are only interested in studying what Feynman says about the density matrix using coupled oscillators. We are particularly interested in studying the effect of Feynman's rest of the universe using the density matrix formalism for the coupled oscillators. In this connection, we note that entropy is the quantity to calculate when we are not able to measure all the variables in quantum mechanics [4,5]. In the present case, we choose the variable x_1 as the world in which we are interested and the x_2 coordinate as the rest of the universe we cannot observe.

Let us translate this into the language of the density matrix:

$$\rho(x_1, x_2; x_1', x_2') = \psi(x_1, x_2)\psi^*(x_1', x_2'), \tag{4.1}$$

for the pure state in the space of both x_1 and x_2. On the other hand, if we do not make measurements in the x_2 space, we have to construct the matrix $\rho(x_1, x_1')$ by integrating over the x_2 variable:

$$\rho(x_1, x_1') = \int \rho(x_1, x_2; x_1', x_2) dx_2. \tag{4.2}$$

It is possible to evaluate the above integral. Since we are now dealing only with the x_1 variable, we shall drop its subscript and use x for the variable for the world in which we are interested. The result of the above integration is

$$\rho(x, x') = \left(\frac{1}{\pi D}\right)^{1/2} \exp\left\{-\frac{1}{4D}\left[(x + x')^2 + (x - x')^2(\cosh^2\eta - \sinh^2\eta\cos^2(2\alpha))\right]\right\}, \tag{4.3}$$

where

$$D = \cosh\eta - \sinh\eta\cos(2\alpha).$$

With this expression, we can check the trace relations for the density matrix. Indeed the trace integral

$$Tr(\rho) = \int \rho(x, x) dx \tag{4.4}$$

becomes 1, as in the case of all density matrices. As for $Tr(\rho^2)$, the result of the trace integral becomes

$$Tr(\rho^2) = \int \rho(x, x')\rho(x', x) dx' dx = \frac{1}{\sqrt{1 + \sinh^2 \eta \sin^2(2\alpha)}}. \tag{4.5}$$

This is less than 1 for non-zero values of α and η. This is consistent with the general theory of density matrices. If $\alpha = 0$ and/or $\eta = 0$, the oscillators become decoupled. In this case, the first oscillator is totally independent of the second oscillator, and the system of the first oscillator is in a pure state, and $Tr(\rho^2)$ becomes 1. This result is also consistent with general theory of density matrices.

We can study the density matrix in terms of the orthonormal expansion given in Eq.(3.4). The pure-state density matrix defined in this case takes the form

$$\rho(x_1, x_2; x_1', x_2') = \sum_{n_1 n_2} \sum_{m_1 m_2} A_{m_1 m_2}(\alpha, \eta) A^*_{n_1 n_2}(\alpha, \eta) \phi_{m_1}(x_1) \phi_{m_2}(x_2) \phi^*_{n_1}(x_1) \phi^*_{n_2}(x_2). \tag{4.6}$$

If x_2 is not measured, and if we use x for x_1, the density matrix $\rho(x, x')$ takes the form

$$\rho(x, x') = \sum_{m,n} \rho_{mn}(\alpha, \eta) \phi_m(x) \phi^*_n(x'), \tag{4.7}$$

with

$$\rho_{mn}(\alpha, \eta) = \sum_k A_{mk}(\alpha, \eta) A^*_{nk}(\alpha, \eta), \tag{4.8}$$

after integrating over the x_2 variable. The matrix $\rho_{mn}(\alpha, \eta)$ is also called the density matrix. The matrix ρ_{mn} is Hermitian and can therefore be diagonalized. If the diagonal elements are ρ_{mm}, the entropy of the system is defined as [4,5]

$$S = -\sum_m \rho_{mm} \ln(\rho_{mm}), \tag{4.9}$$

We measure the entropy in units of Boltzmann's constant k. The entropy is zero for a pure state, and increases as the system becomes impure. Like $Tr(\rho^2)$, this quantity is a measure of our ignorance about the rest of the universe.

In Sec. VI, we shall carry out some concrete calculations for physical problems, including the entropy given in Eq.(4.9). The quantity closely related to the density function is the Wigner phase-space distribution function. In Sec. V, we shall study the effect of Feynman's rest of the universe on the Wigner phase space of the observable world.

V. WIGNER FUNCTIONS AND UNCERTAINTY RELATIONS

In the Wigner phase-space picture of quantum mechanics, both the position and momentum variables are c-numbers. According to the uncertainty principle, we cannot determine the exact point in the phase space of the position and momentum variables, but we can localize these variables into an area in the phase space. The minimum area is of course

Planck's constant. By studying the geometry of this area, we can study the uncertainty relation in more detail than in the Heisenberg or Schrödinger picture. This phase-space representation has been studied extensively in recent years in connection with quantum optics, and there are many books and review articles available on Wigner functions [10]. Thus it is not necessary to give a full-fledged introduction to this subject.

For two coordinate variables, the Wigner function is defined as [10]

$$W(x_1, x_2; p_1, p_2) = \left(\frac{1}{\pi}\right)^2 \int \exp\{-2i(p_1 y_1 + p_2 y_2)\} \times \psi^*(x_1 + y_1, x_2 + y_2)\psi(x_1 - y_1, x_2 - y_2) dy_1 dy_2. \tag{5.1}$$

The Wigner function corresponding to the wave function of Eq.(3.2) is

$$W(x_1, x_2; p_1, p_2) = \left(\frac{1}{\pi}\right)^2 \exp\left\{-e^{\eta}(x_1 \cos\alpha - x_2 \sin\alpha)^2 - e^{-\eta}(x_1 \sin\alpha + x_2 \cos\alpha)^2 - e^{-\eta}(p_1 \cos\alpha - p_2 \sin\alpha)^2 - e^{\eta}(p_1 \sin\alpha + p_2 \cos\alpha)^2\right\}. \tag{5.2}$$

If we do not make observations of the $x_2 p_2$ coordinates and average over them, the Wigner function becomes [11,12]

$$W(x_1, p_1) = \int W(x_1, x_2; p_1, p_2) dx_2 dp_2. \tag{5.3}$$

The evaluation of the integral leads to

$$W(x, p) = \left\{\frac{1}{\pi^2(1 + \sinh^2\eta \sin^2(2\alpha))}\right\}^{1/2} \times \exp\left\{-\left(\frac{x^2}{\cosh\eta - \sin\eta\cos(2\alpha)} + \frac{p^2}{\cosh\eta + \sin\eta\cos(2\alpha)}\right)\right\}, \tag{5.4}$$

where we have replaced x_1 and p_1 by x and p respectively. This Wigner function gives an elliptic distribution in the phase space of x and p. This distribution gives the uncertainty product of

$$(\Delta x)^2(\Delta p)^2 = \frac{1}{4}(1 + \sinh^2\eta \sin^2(2\alpha)). \tag{5.5}$$

This expression becomes 1/4 if the oscillator system becomes uncoupled with $\alpha = 0$. Because x_1 is coupled with x_2, our ignorance about the x_2 coordinate, which in this case acts as Feynman's rest of the universe, increases the uncertainty in the x_1 world which, in Feynman's words, is the system in which we are interested.

Another interesting property of the Wigner function is that it is derivable from the density matrix:

$$W(x_1, x_2; p_1, p_2) = \left(\frac{1}{\pi}\right)^2 \int e^{-2i(p_1 y_1 + p_2 y_2)} \rho\left(x_1 - y_1, x_2 - y_2; x_1 + y_1, x_2 + y_2\right) dy_1 dy_2. \tag{5.6}$$

From this definition, it is straightforward to show [10]

$$Tr(\rho) = \int W(x,p) dx dp, \tag{5.7}$$

and

$$Tr(\rho^2) = 2\pi \int W^2(x,p) dx dp. \tag{5.8}$$

If we compute these integrals, $Tr(\rho) = 1$, as it should be for all pure and mixed states. On the other hand, the evaluation of the $Tr(\rho^2)$ integral leads to the result of Eq.(4.5).

VI. PHYSICAL MODELS

There are many physical models based on coupled harmonic oscillators, such as the Lee model in quantum field theory [13], the Bogoliubov transformation in superconductivity [14], two-mode squeezed states of light [15–19], the covariant harmonic oscillator model for the parton picture [20,21], and models in molecular physics [22]. There are also models of current interest in which one of the variables is not observed, including thermo-field dynamics [23], two-mode squeezed states [11,12], the hadronic temperature [24], and entropy increase caused by Lorentz boosts [25], and the Barnet-Phoenix version of information theory [26]. They are indeed the examples of Feynman's rest of universe. In all of these cases, the mixing angle α is 45^o, and the mathematics becomes much simpler. The Wigner function of Eq.(5.2) then becomes

$$W(x_1,x_2;p_1,p_2) = \left(\frac{1}{\pi}\right)^2 \exp\left\{-\frac{1}{2}\left[e^{\eta}(x_1-x_2)^2 + e^{-\eta}(x_1+x_2)^2 \right.\right.$$
$$\left.\left. + e^{-\eta}(p_1-p_2)^2 + e^{\eta}(p_1+p_2)^2\right]\right\}. \tag{6.1}$$

This simple form of the Wigner function serves as a starting point for many of the theoretical models including some mentioned above.

If the mixing angle α is 45^o, the density matrix also takes a simple form. The wave function of Eq.(3.2) becomes

$$\psi_\eta(x_1,x_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{4}\left[e^{\eta}(x_1-x_2)^2 + e^{-\eta}(x_1+x_2)^2\right]\right\}. \tag{6.2}$$

As was discussed in the literature for several different purposes [10,27,28], this wave function can be expanded as

$$\psi_\eta(x_1,x_2) = \frac{1}{\cosh\eta} \sum_k \left(\tanh\frac{\eta}{2}\right)^k \phi_k(x_1)\phi_k(x_2). \tag{6.3}$$

From this wave function, we can construct the pure-state density matrix

$$\rho(x_1,x_2;x_1',x_2') = \psi_\eta(x_1,x_2)\psi_\eta(x_1',x_2'), \tag{6.4}$$

which satisfies the condition $\rho^2 = \rho$:

$$\rho(x_1,x_2;x_1',x_2') = \int \rho(x_1,x_2;x_1'',x_2'')\rho(x_1'',x_2'';x_1',x_2') dx_1'' dx_2''. \tag{6.5}$$

If we are not able to make observations on the x_2, we should take the trace of the ρ matrix with respect to the x_2 variable. Then the resulting density matrix is

$$\rho(x, x') = \int \psi_\eta(x, x_2) \left\{\psi_\eta(x', x_2)\right\}^* dx_2. \tag{6.6}$$

Here again, we have replaced x_1 and x_1' by x and x' respectively. If we complete the integration over the x_2 variable,

$$\rho(x, x') = \left(\frac{1}{\pi \cosh\eta}\right)^{1/2} \exp\left\{-\left[\frac{(x + x')^2 + (x - x')^2 \cosh^2\eta}{4\cosh\eta}\right]\right\}. \tag{6.7}$$

The diagonal elements of the above density matrix are

$$\rho(x, x) = \left(\frac{1}{\pi \cosh\eta}\right)^{1/2} \exp\left(-x^2/\cosh\eta\right). \tag{6.8}$$

With this expression, we can confirm the property of the density matrix: $Tr(\rho) = 1$. As for the trace of ρ^2, we can perform the integration

$$Tr\left(\rho^2\right) = \int \rho(x, x')\rho(x', x)dx'dx = \frac{1}{\cosh\eta}, \tag{6.9}$$

which is less than one for nonzero values of η.

The density matrix can also be calculated from the expansion of the wave function given in Eq.(6.3). If we perform the integral of Eq.(6.6), the result is

$$\rho(x, x') = \left(\frac{1}{\cosh(\eta/2)}\right)^2 \sum_k \left(\tanh\frac{\eta}{2}\right)^{2k} \phi_k(x)\phi_k^*(x'), \tag{6.10}$$

which leads to $Tr(\rho) = 1$. It is also straightforward to compute the integral for $Tr(\rho^2)$. The calculation leads to

$$Tr\left(\rho^2\right) = \left(\frac{1}{\cosh(\eta/2)}\right)^4 \sum_k \left(\tanh\frac{\eta}{2}\right)^{4k}. \tag{6.11}$$

The sum of this series is $(1/\cosh\eta)$, which is the same as the result of Eq.(6.9).

This is of course due to the fact that we are averaging over the x_2 variable which we do not measure. The standard way to measure this ignorance is to calculate the entropy defined as [5]

$$S = -Tr\left(\rho \ln(\rho)\right), \tag{6.12}$$

where S is measured in units of Boltzmann's constant. If we use the density matrix given in Eq.(6.10), the entropy becomes

$$S = 2\left\{\cosh^2\left(\frac{\eta}{2}\right)\ln\left(\cosh\frac{\eta}{2}\right) - \sinh^2\left(\frac{\eta}{2}\right)\ln\left(\sinh\frac{\eta}{2}\right)\right\}. \tag{6.13}$$

This expression can be translated into a more familiar form if we use the notation

$$\tanh\frac{\eta}{2} = \exp\left(-\frac{\hbar\omega}{kT}\right), \tag{6.14}$$

where ω is given in Eq.(2.13). The ratio $\hbar\omega/kT$ is a dimensionless variable. In terms of this variable, the entropy takes the form

$$S = \left(\frac{\hbar\omega}{kT}\right)\frac{1}{\exp(\hbar\omega/kT) - 1} - \ln\left[1 - \exp(-\hbar\omega/kT)\right]. \tag{6.15}$$

This familiar expression is for the entropy of an oscillator state in thermal equilibrium. Thus, for this oscillator system, we can relate our ignorance to the temperature. It is interesting to note that the coupling strength measured by η can be related to the temperature variable.

VII. CONCLUDING REMARKS

Richard Feynman was a creative physicist and was also a great physics teacher. He had his own way of stating new ideas in physics. Indeed, some of his papers appeared initially strange to the rest of the physics community, such as those on his parton picture [29] and on the relativistic quark model [30]. Yet, his "strange" ideas usually contained deep insights into fundamental problems in physics and opened up new research lines. In this paper, we examined in detail what Feynman may have wanted to say when he introduced the rest of the universe.

It is interesting to note that Feynman's rest of the universe appears as an increase in uncertainty and entropy in the system in which we are interested. In the case of coupled oscillators, the entropy allows us to introduce the variable which can be associated with the temperature. The density matrix is the pivotal instrument in evaluating the entropy. At the same time, the Wigner function is convenient for evaluating the uncertainty product. We can see clearly from the Wigner function how the ignorance or the increase in entropy increases the uncertainty in measurement.

The major strength of the coupled oscillator system is that its classical mechanics is known to every physicist. Not too well known is the fact that this simple device has enough symmetries to serve as an analog computer for many of the current problems in physics. Indeed, this simple system can accommodate the symmetries contained in the group O(3,3) which is the group of Lorentz transformations applicable to three space-like and three time-like dimensions [8]. This group has many interesting subgroups. Many, if not most, of the symmetry groups in physics are subgroups of this O(3,3) group. The coupled oscillators can serve as teaching instruments for many different branches of physics. We are fortunate in this paper to be able to avoid those group theoretical complications.

There are many problems we have not discussed in this paper. If the measurement is less than complete, the result is an increase in entropy. Then the question is whether an increase in entropy leads to a thermal state. This is true for the oscillator problem which we discussed in this paper. However, this is not necessarily true for other cases [25]. It would be an interesting problem to establish a general criterion upon which we can derive the temperature from the density matrix.

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