Generating Fock states and two-Fock states superposition from circular states, in a trapped ion

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Abstract

We propose three schemes to engineer $2^M$ and $M + 1$ circular states for the motion of the center of mass of a trapped ion, $M$ being the number of laser pulses. Since the ion is subjected to several laser pulses, we analyze the necessary duration of each one for generating the circular states, and from these, the Fock states and superposition of two-Fock states. We also calculate the probability for obtaining the required states.

I. INTRODUCTION

An exciting problem in Quantum Mechanics consists in proposing and then generating experimentally states that do not exist in the natural world. Recent advances in quantum optics and atoms manipulation in traps have allowed to realize this challenge with much success. Many nonclassical states have been proposed [1–9] and some have already been produced in QED superconducting cavities and in trapped atoms [10] and ions [11].

Differently from experiments involving atoms interacting with superconducting cavities, trapped ions are loosely coupled to the environment, so decoherence is less manifest, being therefore the best candidates to engineer states or to construct devices as logic gates for implementing quantum computation. However, *decoherence times must be of the order of 1ms* [12].

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Along the line of quantum states engineering Law and Eberly [8] proposed a model where an arbitrary field state in a cavity may be generated by manipulating an atom (source) inside the cavity during the atom-field interaction process, thus all target states (superpositions of Fock number state) can be created from the same initial state. Matos Filho and Vogel [13] proposed a quantum nondemolition measurement of the motional energy of a trapped atom confined in a harmonic potential; by monitoring the interaction times of the lasers interacting with the two-level electronic state of the atom permit to collapse any initial vibrational state into a Fock state, however the Fock state in which the atom collapses after a complete measurement sequence is not predictable. In Refs. [14,15] the authors proposed a general method to entangle quantum states between electronic and vibrational degrees of freedom in order to engineer a vibrational (trapped ion) target state out from an initial vacuum one; they also analyzed the sensitivity of the method due to errors in the amplitude and phase of the lasers. Simulation for preparation of Fock states by observation of quantum jumps were reported in Ref. [16]. See also [17] for the synthesis of an arbitrary two-mode bosonic state.

More subtly, Fock states can also be obtained by a superposition of coherent states evenly distributed on a circle in the phase space called circular states. Experimental schemes were proposed to generate such superpositions as states of the electromagnetic (EM) field in superconducting cavities [3,5–7,18] and also as states of the harmonic motion of the center of mass (COM) of a trapped ion [19,20]. Recently, a procedure to create an arbitrary coherent states superposition of the vibrational motion of a trapped ion, localized along a line in phase space representation (target state) was proposed [21].

Here we analyze three schemes to engineer circular states (CS) as target states for the COM of a trapped ion, estimate the total time of laser pulses and the probability to produce such states. This contribution is organized as follows: In section II we present a brief revision of the circular states and establish the conditions to obtain as a result of the $N$ coherent states interference in phase space: the vacuum state, Fock states, special superpositions of two-Fock states. In section III we review the ion-laser interaction process. In section IV we estimate the necessary duration of the laser pulses to generate the wanted states and the probability for producing them. Finally in section V we present a summary and conclusions.

II. CIRCULAR STATES, FOCK STATES AND TWO-FOCK STATES

The circular state, proposed and studied by Janszky, coworkers [4,5,22] and Gagen [23], generalizes the Schrödinger cat state. It consists of a superposition of $N$ coherent states $|\alpha_k\rangle$ equally likely distributed on a circle of radius $r = |\alpha_k|$ in the phase space defined by the c-number $\alpha_k$,

$$|\Psi_N(\alpha_0)\rangle = \lambda_N^{-1/2} \sum_{k=1}^{N} C_k |\alpha_k\rangle \quad \text{with} \quad \alpha_k = \alpha_0 e^{\frac{2\pi i k}{N}},$$

(1)

where $\lambda_N^{-1/2}$ is the normalization constant. For $N$ even and $C_{k+N/2} = \pm C_k$, the circular state is even or odd: $|\Psi_N(-\alpha_0)\rangle = \pm |\Psi_N(\alpha_0)\rangle$ and $\hat{a}_N^N |\Psi_N(\alpha_0)\rangle = \alpha_0^N |\Psi_N(\alpha_0)\rangle$. Hereon we assume $N$ even and consider two kinds of CS's, an even and an odd:

(a) For $C_k = 1$, the circular state is even.
\(|\Psi_N\rangle = \lambda_N^{-1/2} \sum_{k=1}^{N} |\alpha_k\rangle, \quad (2)
\]

with normalization factor

\[
\lambda_N = \left[ N + 2 \sum_{k=1}^{N-1} ke^{-2r^2\sin^2(\frac{k\pi}{N})} \cos \left( r^2 \sin \left( \frac{2\pi k}{N} \right) \right) \right].
\]

In the Fock states basis state (2) is

\[
|\Psi_N\rangle = Z_N^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha_0^n N^n}{\sqrt{N!n!}} |Nn\rangle = \frac{|0\rangle + \frac{r^N}{\sqrt{N!}} |N\rangle + \frac{r^{2N}}{\sqrt{(2N)!}} |2N\rangle + \cdots}{\sqrt{1 + \frac{r^{2N}}{N!} + \frac{r^{4N}}{(2N)!} + \cdots}},
\]

with normalization constant

\[
Z_N (r^2) = \sum_{n=0}^{\infty} \frac{(r^2)^n N^n}{(Nn)!}.
\]

Due to interference between the coherent states the circular states may emulate essentially one Fock state or a superposition of two-Fock states under the following conditions:

\[
\begin{cases}
1 \ll (er^2/N)^N \ll 4^N & \text{gives closely a Fock state } |N\rangle \\
(er^2/N)^N \ll 1 & \text{gives closely the vacuum state } |0\rangle \\
e^{-2}/N \simeq 1 & \text{gives closely the superposition } |0\rangle + |N\rangle \\
e^{-2}/(4N) \simeq 1 & \text{gives closely the superposition } |N\rangle + |2N\rangle.
\end{cases}
\]

The probability for a circular state having exactly \(n\) photons is

\[
P_n \equiv |\langle n|\Psi_N(\alpha_0)\rangle|^2 = Z_N^{-1} \frac{r^{2n}}{n!} \delta_{n,Nk} = P_{Nk} (r^2), \quad (k = 0, 1, 2, \ldots; N = 2, 3, 4, \ldots). \quad (3)
\]

\((N = 1\) is trivially the coherent state) meaning that for fixed value of \(N\), the circular state has a finite probability for 0, \(N\), 2\(N\), 3\(N\), ... quanta and zero probability for any other number of quanta. The normalization constant \(Z_N (r^2)\) is also the partition function of the statistical distribution of quanta, thus the mean value of any power of the number of quanta is given by

\[
\langle \hat{n}^k \rangle = \frac{1}{Z_N (y)} \left( y \frac{\partial}{\partial y} \right)^k Z_N (y), \quad y = r^2.
\]

(b) For \(C_k = e^{i2\pi k/N}\) the circular state is odd,

\[
|\bar{\Psi}_N\rangle = \lambda_N^{-1/2} \sum_{k=1}^{N} e^{\frac{2\pi ik}{N}} |\alpha_k\rangle, \quad (4)
\]

with normalization factor

\[
\lambda_N = \left[ N + 2 \sum_{k=1}^{N-1} ke^{-2r^2\sin^2(\frac{k\pi}{N})} \cos \left( r^2 \sin \left( \frac{2\pi k}{N} \right) \right) \right].
\]
\[ \lambda_N = \left[ N + 2 \sum_{k=1}^{N-1} ke^{-2r^2 \sin^2 \left( \frac{\pi k}{N} \right)} \cos \left( \frac{2\pi k}{N} + r^2 \sin \left( \frac{2\pi}{N} k \right) \right) \right]. \] (5)

In terms of Fock states it can be written as

\[ \tilde{\Psi}_N = Z_N^{-1/2} \sum_{k=1}^{\infty} \frac{r^{(kN-1)}}{\sqrt{(kN-1)!}} |kN - 1\rangle = \frac{|N - 1\rangle + r^N \left( \frac{(N-1)!}{(2N-1)!} \right)^{1/2} |2N - 1\rangle + ...}{\sqrt{1 + r^2N \left( \frac{(N-1)!}{(2N-1)!} \right)^2 + ...}}, \] (6)

where the only Fock states present in the superposition are \( N - 1, 2N - 1, 3N - 1, \ldots, kN - 1 \) \((k = 1, 2, \ldots; N = 2, 3, \ldots)\) and when \((r^2e/4N)^N \ll 1\) only the first Fock state \(|N - 1\rangle\) is important.

The partition function is

\[ Z_N (r^2) = \sum_{k=1}^{\infty} \frac{r^{2(kN-1)}}{(kN-1)!}, \]

and the probability for each Fock state \(|n\rangle\) to be present in the superposition is

\[ P_n (r^2) \equiv \left| \langle n | \tilde{\Psi}_N \rangle \right|^2 = \lambda_N^2 \frac{r^{2(kN-1)}}{(kN-1)!} \delta_{n,kN-1} = P_{kN-1} (r^2). \] (7)

In Table 1 we present a summary of our results, where \(a = r^2\), and for each pair \((N, a)\), \(P_{N,j}\) corresponds to the overlap between the Fock state (or two-Fock states superposition) and the CS. \(P_1 (a)\) is the probability for the experimental production of a particular state for the ion COM vibrational motion. \(\bar{n}\) and \(\text{Var}(\bar{n})\) stand for the mean numbers and variances of quanta.

<table>
<thead>
<tr>
<th>Special Even Circular States</th>
<th>((N, a))</th>
<th>(P_{N,j})</th>
<th>(P_1 (a))</th>
<th>(\bar{n})</th>
<th>(\text{Var}(\bar{n}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Psi_{8,0} \approx (</td>
<td>0\rangle +</td>
<td>8\rangle) / \sqrt{2})</td>
<td>((8, 3.76))</td>
<td>0.99996</td>
<td>0.04635</td>
</tr>
<tr>
<td>(\Psi_8^{(1)} \approx</td>
<td>8\rangle)</td>
<td>((8, 6.80))</td>
<td>0.982674</td>
<td>0.12851</td>
<td>7.99994</td>
</tr>
<tr>
<td>(\Psi_{8,1} \approx (</td>
<td>8\rangle +</td>
<td>16\rangle) / \sqrt{2})</td>
<td>((8, 12.17))</td>
<td>0.991848</td>
<td>0.12016</td>
</tr>
<tr>
<td>(\Psi_8^{(2)} \approx</td>
<td>16\rangle)</td>
<td>((8, 15.80))</td>
<td>0.79002</td>
<td>0.12542</td>
<td>15.99847</td>
</tr>
<tr>
<td>(\Psi_{16,0} \approx (</td>
<td>0\rangle +</td>
<td>16\rangle) / \sqrt{2})</td>
<td>((16, 6.80))</td>
<td>0.99999</td>
<td>0.00261</td>
</tr>
<tr>
<td>(\Psi_{16}^{(1)} \approx</td>
<td>16\rangle)</td>
<td>((16, 12.80))</td>
<td>0.999918</td>
<td>0.08564</td>
<td>16.00001</td>
</tr>
<tr>
<td>(\Psi_{16,1} \approx (</td>
<td>16\rangle +</td>
<td>32\rangle) / \sqrt{2})</td>
<td>((16, 24.10))</td>
<td>0.99807</td>
<td>0.04295</td>
</tr>
<tr>
<td>(\Psi_{16}^{(2)} \approx</td>
<td>32\rangle)</td>
<td>((16, 31.20))</td>
<td>0.96834</td>
<td>0.07190</td>
<td>32.02314</td>
</tr>
</tbody>
</table>

Table 1. The first column stands for the particular states coming out from a \(N\)-even CS, due to interference in phase space; the second column gives the corresponding pairs of values \((N, a)\) \((a = r^2)\) to be attributed to the CS; the third column stands for the maximum probability of a particular state in the CS superposition; the fourth column shows the probability for the experimental production of a particular state for the ion COM vibrational motion; finally, fifth and sixth columns give the corresponding mean numbers and variances.
III. ION-LASER INTERACTION

A trapped ion is considered moving in a one-dimensional harmonic effective pseudopotential, interacting with two laser (frequencies $\omega_1$ and $\omega_2$) in a Raman-type configuration, which is responsible for a forbidden transition between two metastable internal electronic states, $|\uparrow\rangle$ and $|\downarrow\rangle$, separated by frequency $\omega_0$, and called excited and ground, respectively. In this configuration, the two levels are indirectly coupled via a third one, $|r\rangle$, which is adiabatically eliminated. A fourth level $|d\rangle$ is used to cool the ion and to measure its internal electronic state by fluorescence emission, moreover it can still be used to generate the ion motional states.

The Hamiltonian describing the effective interaction between the two-level electronic states and the quantized motion of the ion COM is written as (we consider $\hbar = 1$):

$$H = \omega a^\dagger a + \frac{\delta}{2} \sigma_z - \Omega \left( \sigma_- e^{-i\eta(a^\dagger a)^{1/2}} + \sigma_+ e^{i\eta(a^\dagger a)^{1/2}} \right),$$

where $\sigma_+ (\sigma_-) = |\uparrow\rangle\langle \downarrow| (|\downarrow\rangle\langle \uparrow|)$ and $\sigma_z$ are the usual Pauli pseudospin operators, $a^\dagger (a)$ is the creation (annihilation) operator of vibrational quanta, $\Omega$ is the effective Rabi frequency of the transition $|\uparrow\rangle \leftrightarrow |\downarrow\rangle$, $\omega$ is the ion vibrational frequency, $\delta = \omega_1 - \omega_2 - \omega_0$ is the detuning frequency and

$$\eta = \Delta k / \sqrt{2m\omega},$$

is the Lamb-Dicke parameter, where $\Delta k = (\vec{k}_1 - \vec{k}_2) \cdot \hat{x}$ and $|\vec{k}_{1(2)}\rangle = \omega_{1(2)}/c$, being $\vec{k}_1 (\vec{k}_2)$ the wave vector of laser 1(2), $\hat{x}$ is the operator referring to the COM position of the ion, $m$ is its mass and $\omega$ is the angular frequency of its (very closely) harmonic oscillation in the trap. Thus, the strength of $\eta$ can be arbitrarily selected by changing the relative direction of the laser beams.

Writing first $H$ in the interaction picture and then expanding the resulting Hamiltonian in terms of the Lamb-Dicke parameter one gets [24]

$$H_I = -\Omega e^{-\eta^2/2} \left[ \sum_{m,l=0}^{\infty} \frac{(i\eta)^{m+l}}{m!l!} a^\dagger_m a^l e^{i[(m-l)\omega+\delta]t+i\phi} \sigma_- + h.c. \right].$$

Since the frequency $\omega$ is large in comparison with $\Omega$, resonance conditions result for $\delta = -k\omega$ ($k = m - l$).

(a) The choice $k = 0$ selects the carrier transition, with Hamiltonian

$$H_I^{(c)} = -\Omega f_0(\hat{n}) \left[ \sigma_+ e^{-i(\Delta t-\phi)} + \sigma_- e^{i(\Delta t-\phi)} \right]$$

(b) $k > 0$ selects the $k$-th red frequency, whose Hamiltonian is

$$H_I^{(r)} = -\Omega \left[ f_k(\hat{n}) \sigma_+ a^k e^{-i(\Delta t-\phi)} + f_k^\dagger(\hat{n}) \left( a^\dagger \right)^k \sigma_- e^{i(\Delta t-\phi)} \right]$$

(c) $k < 0$ ($k = -k'$) selects the $k$-th blue frequency, with Hamiltonian
\[ H^{(b)}_{1,k'} = -\Omega \left[ (a^{\dagger})^k f_k(\hat{n}) \sigma_+ e^{-i(\Delta t - \phi)} + f_{k'}(\hat{n}) a^{k'} \sigma_- e^{i(\Delta t - \phi)} \right] \]

where \( f_k(\hat{n}) = e^{-\eta^2/2} \sum_{l=0}^{\hat{n}} \frac{(i\eta)^{2l+k}}{\sqrt{l!(l+k)!}} (\hat{n} - l)! \). So, engineering a particular quantum state becomes possible by choosing a particular hamiltonian from Eq. (10). Considering the Lamb-Dicke limit, \( \eta^2 \ll 1 \), some specific hamiltonians have been proposed and investigated, namely: the Jaynes-Cummings hamiltonian \( (k = 1) \), the anti-Jaynes-Cummings hamiltonian \( (k' = 1) \) and the carrier hamiltonian \( (k = 0) \), which couple, respectively, the levels \(|m + 1 \downarrow\rangle \leftrightarrow |m \uparrow\rangle\), \(|m \downarrow\rangle \leftrightarrow |m + 1 \uparrow\rangle\) and \(|m \downarrow\rangle \leftrightarrow |m \uparrow\rangle\).

The measurement of the ion vibrational state is achieved indirectly through the measurement of the ion electronic state which is realized by collecting the resonance fluorescence signal from the transition \(|d\rangle \leftrightarrow |\downarrow\rangle\) through another laser strongly coupled to the electronic ground state. The measured signal is the probability of the ion to be found in the internal state \(|\downarrow\rangle\). Since the fluorescence emission disturbs the COM motion of the ion, for each new measurement the ion should be cooled back to its ground state.

IV. GENERATION OF CIRCULAR STATES IN A TRAPPED ION

We propose and analyze three different schemes to generate circular states of the kind of Eqs. (2) and (4). For this purpose we consider a type-Kerr interaction between the ion and the effective laser which is realized by tuning it resonantly \( (\delta = 0) \) with the electronic transition frequency \( \omega_0 \). In the Lamb-Dicke limit, the carrier hamiltonian becomes

\[ H = \Omega \sigma_x - \eta^2 \Omega \left[ \left( 1 + \frac{\eta^2}{4} \right) a^{\dagger}a - \frac{\eta^2}{4} \left( (a^{\dagger})^2 + O(\eta^4) \right) \right] \sigma_x \approx \Omega \sigma_x - \eta^2 \Omega a^{\dagger}a \sigma_x, \]

(setting \( \phi = \pi \) without lost of generality). The approximation is valid under the condition

\[ \eta^2 (Q + \bar{n} + 1)/4 \ll 1, \]

(12)

where, \( Q \equiv (\text{Var}(\hat{n}) - \bar{n})/\bar{n} \) is the Mandel Q-parameter, \( \bar{n} \) is the mean number of the ion motional quanta and \( \text{Var}(\hat{n}) \) is the variance. For instance, if the ion is prepared initially in a coherent state \(|\alpha_0\rangle\), then (12) becomes \((|\alpha_0|^2 + 1)\eta^2/4 \ll 1\), which must be satisfied in order to consider the approximation in Eq. (11). Thus, assuming condition (12) that allow the use of (11), the evolution operator is

\[ U(t) \equiv e^{-iHt} = e^{-i\bar{\Omega}t \sigma_x} e^{i\bar{\Omega}t \sigma_x a^{\dagger}a}, \]

(13)

where \( \bar{\Omega} \equiv \eta^2 \Omega \). In all three schemes the ion is initially prepared in the upper electronic state \(|\uparrow\rangle\) and its COM is in a coherent state,

\[ |\Psi(0)\rangle \equiv |\alpha_0\rangle \otimes |\uparrow\rangle = |\alpha_0\rangle \otimes \frac{|\uparrow_x\rangle + |\downarrow_x\rangle}{\sqrt{2}}, \]

(14)

the second equality follows because in terms of the eigenvectors \(|\uparrow\rangle, |\down\rangle\) of \( \sigma_z \), one has \(|\uparrow_x\rangle = (|\uparrow\rangle + |\down\rangle)/\sqrt{2} \) and \(|\down_x\rangle = (|\uparrow\rangle - |\down\rangle)/\sqrt{2}\).
The procedure is the following: one applies \(M\) laser pulses, each with duration \(t_k\), chosen according to the state one wants to generate. The \(M\) pulses are triggered in sequence as long as one observes \(M\) no-fluorescence events when measuring the \(|d⟩ \leftrightarrow |↓⟩\) transition. If in one of the measurement the ion is found in the ground electronic state \(|↓⟩\) (fluorescence is observed) the process is stopped and one should repeat the sequence of pulses after preparing again the ion in a coherent state. The no-fluorescence measurement is a necessary condition because it assures the no-recoil of the ion vibrational COM motion, thus, maximizing the probability of this particular sequence be realized successfully.

A. First Scheme - \(2^M\) even circular states, with \(M\) laser pulses

Our aim is to generate a circular state \((2)\) with \(N = 2^M\) coherent states. We have to perform two operations repeatedly on the ion with the lasers, (1) evolve unitarily the ion state with \(U(t)\) \((13)\) and \(2\) do a projection of the ion statevector on \(|↑⟩\) in order to measure the probability to be found in the upper electronic state. So, in this scheme each cycle involves the application of one pulse and one measurement. The first pulse having duration \(t_1\), the ion state becomes

\[
|Ψ(t_1)⟩ = U(t_1) |Ψ(0)⟩ = \frac{1}{2} \left( e^{-iΩt_1} |α(t_1)⟩ + e^{iΩt_1} |α(-t_1)⟩ \right) ⊗ |↑⟩
\]

\[
+ \frac{1}{2} \left( e^{-iΩt_1} |α(t_1)⟩ - e^{iΩt_1} |α(-t_1)⟩ \right) ⊗ |↓⟩,
\]

where \(α(t) ≡ |α_0e^{iΩt}⟩\). The evolution is followed by a fluorescence measurement, if the ion is found in state \(|↑⟩\) (no fluorescence) we consider it a successful measurement, then state \((15)\) reduces to

\[
|Ψ'(t_1)⟩ = \frac{N_1}{2} \left( e^{-iΩt_1} |α(t_1)⟩ + e^{iΩt_1} |α(-t_1)⟩ \right) ⊗ |↑⟩,
\]

where \(N_1\) is the normalization factor. The probability to find the ion in state \(|↑⟩\) is

\[
P_1(t_1) = \frac{1}{2} \left( 1 + e^{-r^2(1-\cos[2Ωt_1])} \cos \left[ r^2 \sin \left[ 2Ωt_1 - 2Ωt_1^2 \right] \right] \right).
\]

Assuming the ion was measured in this state, a second laser pulse drives the ion into state \(|Ψ(t_1 + t_2)⟩ = U(t_2) |Ψ'(t_1)⟩\) and the second successful measurement reduces this state to

\[
|Ψ'(t_1 + t_2)⟩ = \frac{N_2}{4} \left( e^{-iΩ(t_1+t_2)} |α(t_1 + t_2)⟩ + e^{-iΩ(t_1-t_2)} |α(t_1 - t_2)⟩ \right)
\]

\[
+ e^{iΩ(t_1-t_2)} |α(t_2 - t_1)⟩ + e^{-iΩ(t_1+t_2)} |α(-t_1 - t_2)⟩ \right) ⊗ |↑⟩,
\]

with probability

\[
P_1(t_1 + t_2) = \frac{1}{4} \left( 1 + e^{-r^2(1-\cos[2Ωt_1])} \cos \left[ r^2 \sin \left[ 2Ωt_1 - 2Ωt_1^2 \right] \right] \right.
\]

\[
+ e^{-r^2(1-\cos[2Ωt_2])} \cos \left[ r^2 \sin \left[ 2Ωt_2 - 2Ωt_2^2 \right] \right]
\]

\[
+ e^{-r^2(1-\cos[2Ω(t_1+t_2)])} \cos \left[ r^2 \sin \left[ 2Ω (t_1 + t_2) \right] - 2Ω (t_1 + t_2) \right]
\]

\[
+ e^{-r^2(1-\cos[2Ω(t_1-t_2)])} \cos \left[ r^2 \sin \left[ 2Ω (t_1 - t_2) \right] - 2Ω (t_1 - t_2) \right]\right).
\]
and so on.

In order to engineer the ion vibrational state as Eq. (2) we need to adjust the phases of the coherent states such to be evenly distributed on the circle, this is possible when the duration of the \(k\text{-th}\) pulse is

\[ t_k = \frac{\pi}{2k\Omega}. \]  

(20)

Besides, we remind that all the coefficients of superposition (18), for instance, should be equal to 1, so choosing the Lamb-Dicke parameter as \(\eta^2 = 2^{-(M+1)}\), it becomes possible to generate the target superposition state.

The probability to get the ion successfully in the upper electronic state after each of the \(M\) sequential measurements is given by

\[ P_1 \left( \sum_{k=1}^{M} t_k \right) = \frac{1}{2^M} \left\{ 1 + \frac{1}{2^{M-1}} \sum_{k=1}^{2^{M-1}} k \exp \left[ -2r^2 \sin^2 \left( \frac{\pi k}{2M} \right) \cos \left[ r^2 \sin \left( \frac{\pi k}{2M-1} \right) \right] \right] \right\}. \]  

(21)

In the coherent states \(|\alpha_k\rangle\) the c-numbers \(\alpha_k\) have the following phases distributed on the circle of radius \(r\)

\[ \theta_k^{(\pm)} = \theta_0 \pm \frac{\pi}{2M} (2k - 1) \quad k = 1, \ldots, 2^{M-1}. \]  

(22)

For example, if one wants to engineer a superposition with 16 coherent states \((M = 4\text{ is the number of cycles})\) we need \(\eta \approx 0.18\). Proceeding along the lines drawn above we get the following states after each successful measurement:

\[ t_1 = \frac{\pi}{2\Omega} \Rightarrow |\Psi' (t_1)\rangle = \mathcal{N}_1 \left( |\alpha_0 e^{i\pi/2}\rangle + |\alpha_0 e^{-i\pi/2}\rangle \right)/2 \]

\[ t_2 = \frac{\pi}{4\Omega} \Rightarrow |\Psi' (t_1 + t_2)\rangle = \mathcal{N}_2 \left( |\alpha_0 e^{3i\pi/4}\rangle + |\alpha_0 e^{i\pi/4}\rangle + |\alpha_0 e^{-i\pi/4}\rangle + |\alpha_0 e^{-3i\pi/4}\rangle \right)/4 \]

\[ t_3 = \frac{\pi}{8\Omega} \Rightarrow |\Psi' (t_1 + t_2 + t_3)\rangle = \mathcal{N}_3 \left( |\alpha_0 e^{7i\pi/8}\rangle + |\alpha_0 e^{5i\pi/8}\rangle + |\alpha_0 e^{3i\pi/8}\rangle + |\alpha_0 e^{i\pi/8}\rangle + |\alpha_0 e^{-i\pi/8}\rangle + |\alpha_0 e^{-3i\pi/8}\rangle + |\alpha_0 e^{-5i\pi/8}\rangle + |\alpha_0 e^{-7i\pi/8}\rangle \right)/8, \]

etc. So, it suffices to control the on-off switchings of the laser beams to get the right state. The total time necessary for the \(M\) pulses is

\[ T = \sum_{k=1}^{M} t_k = \frac{\pi}{\eta^2\Omega} \left( 1 - \frac{1}{2^M} \right) = 2^{M+1}\pi \frac{1}{\Omega} (1 - \frac{1}{2^M}) = \frac{2\pi}{\Omega} (2^M - 1) \approx (2^M - 1)\mu s, \]  

(23)

since \(2\pi/\Omega \approx 1\mu s\), so about \(15\mu s\) is the required duration of pulses to get a superposition of 16 coherent states, this time is much less than that necessary for doing the experiment proposed in [14].

The probability for the first successful cycle is \(P_1(t_1) = \left[ 1 + e^{-2r^2} \right]/2\) while for the second successful cycle is \(P_1(t_1 + t_2) = \frac{1}{4} \left[ 1 + e^{-2r^2} + 2e^{-r^2} \cos r^2 \right]\) and so forth.
Since the times \( t_k \) are already fixed we only have the freedom to choose the radius \( r \) (the intensity of initial coherent state) and the number of pulses to engineer a particular state. An example: For \( 2^4 = 16 \) superposed coherent states and \( r = 3.6 \) the probability for producing approximately a Fock state \(|16\rangle\) is \( \approx 0.09 \), which is not a bad result since in the average one in eleven runs is successful, in this case both conditions \( 1 \ll \left(\frac{er^2}{2^4}\right)^2 \ll 4^{16} \) and \( r^2 \ll 8(16) - 1 \) are satisfied. See the fourth column of Table 1 for further results.

**B. Second scheme - \( N = M + 1 \) even circular states**

Now, we will describe a scheme that generates a superposition of \( M+1 \) arbitrary coherent states on the circle as a result of selecting all pulses having the same duration \( \tau \).

If we consider equal times \( \tau \) in Eqs. (15) and (18), after two successful cycles we get

\[
|\Psi'(\tau)\rangle = \frac{N_1}{2} (|\alpha(\tau)\rangle + |\alpha(-\tau)\rangle) \otimes |\uparrow\rangle, \\
|\Psi'(2\tau)\rangle = \frac{N_2}{4} (|\alpha(2\tau)\rangle + 2|\alpha(0)\rangle + |\alpha(-2\tau)\rangle) \otimes |\uparrow\rangle, \tag{24}
\]

which is not what we need because the second term has a factor 2 and we want all the coefficients in (24) be equal to 1. Thus, we have to use a second pair of lasers (also in the Lamb-Dicke limit) with Lamb-Dicke parameter \( \eta_r \) satisfying the condition \( \eta^2 \Lambda \ll \eta^2 \Omega \) in order to rotate the electronic levels according to the evolution ruled by

\[
H_r = \Lambda \sigma_x \Rightarrow U_r(t) = e^{-i\Lambda t \sigma_x}. \tag{25}
\]

Now, each cycle consists of one rotating pulse, one evolution pulse and one measurement.

The action of the \( l \)-th rotating pulse on \(|\uparrow\rangle\) with duration \( t'_l \) is

\[
U_r(t'_l)|\uparrow\rangle = \frac{e^{-i\Lambda t'_l}}{\sqrt{2}} |\uparrow_x\rangle + \frac{e^{i\Lambda t'_l}}{\sqrt{2}} |\downarrow_x\rangle = \frac{a_l}{\sqrt{2}} |\uparrow_x\rangle + \frac{b_l}{\sqrt{2}} |\downarrow_x\rangle. \tag{26}
\]

Assuming each cycle is successful (no-fluorescence measurement), the evolution of the ion vibrational state after, for instance, \( M = 3 \) cycles goes as follow (\( \Omega \tau = 2\pi \)):

\[
|\alpha\rangle \rightarrow a_1 |\alpha(\tau)\rangle + b_1 |\alpha(-\tau)\rangle \rightarrow \\
a_1a_2 |\alpha(2\tau)\rangle + (a_1b_2 + b_1a_2) |\alpha(0)\rangle + b_1b_2 |\alpha(-2\tau)\rangle \rightarrow \\
a_1a_2a_3 |\alpha(3\tau)\rangle + (a_1a_2b_3 + a_1b_2a_3 + b_1a_2a_3) |\alpha(\tau)\rangle + \\
(a_1b_2b_3 + b_1a_2b_3 + b_1b_2a_3) |\alpha(-\tau)\rangle + b_1b_2b_3 |\alpha(-3\tau)\rangle. \tag{27}
\]

In order to adjust the coefficients in (27) to reproduce the phases of \( \alpha_k \) in Eq. (2), we need first to attribute values to the duration of each evolution pulse and to the Lamb-Dicke parameter, \( \tau = \frac{\pi}{(M+1)\Omega} \), \( \eta = \frac{1}{\sqrt{2(M+1)}} \).

For determining the time intervals \( t'_l \) of each rotating pulse we set each coefficient in (27) equal to 1, so we need to solve a system of algebraic equations. Calling \( z_l = b_l/a_l \), the system of equations is
\[ z_1 z_2 z_3 = 1, \quad z_1 + z_2 + z_3 = 1, \quad z_1 z_2 + z_2 z_3 + z_1 z_3 = 1, \]

whose solution are the roots of the polynomial equation \( z^3 - z^2 + z - 1 = 0 \). In general, after \( M \) successful cycles the solution of the equation

\[ \sum_{k=0}^{M} (-1)^{M-k} z^k = 0, \tag{28} \]

are the roots \( z_l = e^{i\pi + 2\pi i l/(M+1)} \), \( l = 1, 2, ..., M \). But since \( z_l = e^{2\pi i l'/M} \), so the \( l \)-th pulse time interval must be \( t'_l = \left( \frac{l}{M+1} + \frac{1}{2} \right) \frac{\pi}{M} \). The probability to produce such a state is

\[
P_\uparrow \left( M \tau + \sum_{l=1}^{N} t'_l \right) = \frac{1}{2^M} \left\{ M + 1 + 2 \sum_{k=1}^{M} k \exp \left[ -2r^2 \sin^2 \left( \frac{\pi k}{M+1} \right) \right] \right\},
\]

and the total time of pulses is \( T = M \tau + \sum_{l=1}^{N} \left( \frac{l}{M+1} + \frac{1}{2} \right) \frac{\pi}{M} = M \left( \frac{2}{M+1} + \frac{\pi}{M} \right) \). One verifies that for a large \( M \), \( P_\uparrow \left( M \tau + \sum_{l=1}^{M} t'_l \right) \) is very small and \( T \) is very large. An example: for engineering \( M+1 = 16 \) superposed states one needs 15 cycles, with \( P_\uparrow (t_1 + ... + t_{15}) \approx 1/2^{15} \); the total duration of the pulses is \( T = 15 \left( 2\pi/\Omega + \pi/\Lambda \right) \), exceeding the total time of the previous scheme by \( 15 \pi/\Lambda \) which is large because experimental conditions imply \( \Lambda \ll \Omega \) since \( \eta_r \ll \eta \). Thus it is highly unlikely to generate a Fock state for \( N > 4 \) since the probability is quite small, whereas for a small \( M \) we remind that the condition \( (e r^2/(M + 1))^{M+1} \gg 1 \) must be fulfilled.

C. Third scheme - \( N = 2^M \) odd circular states

To construct the superposition of \( 2^M \) coherent states of odd parity (4) we have to combine the two previous schemes, since now we need one rotating pulse, one evolution pulse and one measurement per cycle. A pair of lasers in-phase is used to produce a rotating pulse, such that \( H = -\Lambda \sigma_x \).

The evolving and rotating pulses have duration \( t_k = \pi / \left( 2^k \bar{\Omega} \right) \) and \( t'_l = \pi / \left( 2^l \Lambda \right) \), respectively, and \( \eta_r \ll \eta \). After \( M \) successful cycles the ion state becomes

\[
\left| \tilde{\Psi} \left( \sum_{k=1}^{M} t_k + \sum_{l=1}^{M} t'_l \right) \right> = \frac{N}{2^M} \left| \tilde{\Psi}_{2^M} \right> \otimes |\uparrow> ,
\]

with probability

\[
P_\uparrow \left( \sum_{k=1}^{M} t_k + \sum_{l=1}^{M} t'_l \right) = \frac{1}{2^M} \left\{ 1 + 2^{-(M-1)} \sum_{k=1}^{2^M-1} k \exp \left[ -2r^2 \sin^2 \left( \frac{\pi k}{2^M} \right) \right] \right\} \times \cos \left[ 2\pi k / 2^M + r^2 \sin \left( 2\pi k / 2^M \right) \right] .
\]

(31)
and the total duration of the pulses is $T = \left(2^M - 1\right)\left(\frac{2\pi}{w} + \frac{\pi}{2^M\Lambda}\right)$. We remind that $\left(r^2e/2^{M+2}\right)^{2^M} \ll 1$ is the necessary condition for generating approximately a Fock state $\lvert 2^M - 1 \rangle$ from odd parity circular state ($2^M$ coherent states).

An example: for $M = 4$ the probability is maximum, $P \left(\sum_{k=1}^{4} t_k + \sum_{l=1}^{4} t_l\right) \approx 0.1$, for $r = 4$, thus generating approximately the Fock number state $\lvert 15 \rangle$. However, the total time involved in the engineering of the state $\tilde{\Psi} \left(\sum_{k=1}^{4} t_k + \sum_{l=1}^{4} t_l\right) \approx \lvert 15 \rangle$ is $T = 15\left(2\pi/\Omega + \pi/16\Lambda\right)$, which is larger than in the first scheme by $15\pi/16\Lambda$, showing that it takes more pulses time to generate an odd Fock number state than an even one.

V. CONCLUSIONS

We have analyzed three schemes to engineer particular circular states for the vibrational mode of a trapped ion. We have considered the ion interacting with two laser beams in a stimulated Raman configuration and have selected the effective laser frequency in resonance with the ion transition electronic frequency, which resulted into an effective interaction of type-Kerr in the ion-laser system. After preparing the system in a particular initial state ($\lvert \alpha_0 \uparrow \rangle$) we have considered $M$ operations where each successful cycle results in a new (extended) circular state for the vibrational motion. We have calculated the total time of pulses necessary for each kind of circular state and the probability to produce it. We have shown the possibility to construct Fock states and two-Fock states superpositions out from the circular states and verified that it takes much more pulses time to generate an odd circular state $\lvert 2^M - 1 \rangle$ than the even $\lvert 2^M \rangle$.

The measurement process occurs when the laser is coupled to the $\lvert d \rangle \leftrightarrow \lvert \downarrow \rangle$ transition, where the width of the $\lvert d \rangle$ level is $\Gamma$ and $\Gamma/2\pi \approx 20$ MHz ($2\pi/\Gamma \approx 0.05 \mu$ s), the duration of the pulse $T'$ must be sufficiently long to allow the emission of at least one photon with high probability. The non-observation of a fluorescence photon during time $T'$ is a measurement, meaning that the internal electronic state is in level $\lvert \uparrow \rangle$, thus collapsing the state, and so the process of the other lasers pulses can proceed. According to [14] $T' \ll 2\mu$s, assuming a $T' \approx 0.2 \mu$s, the time $NT'$, where $N$ is the number of cycles of pulses, is to be added to the pulses total time for producing successfully a Fock number state. At much it will represent about or less than 10% of the total time.

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